QUANTITATIVE STABILITY FOR MINKOWSKI'S PROBLEM

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Abstract. We derive quantitative stability results for Minkowski bodies, as well as their counterparts, the L_p-Minkowski bodies in the range $1 \leq p \neq n$. We prove that, for every pair of probability measures μ, ν satisfying a quantitative form of the classical dispersion assumptions yielding existence of such bodies, we have a control of the form

$$\alpha(E_\mu, E_\nu)^2 \leq C\, \mathrm{d_C}(\mu,\nu)^{1+\frac{1}{n}},$$

where α denotes the Fraenkel asymmetry and $\mathrm{d_C}$ is the dual-convex distance of probability measures on the sphere. Our arguments are based on a variational problem whose optimizers are Minkowski bodies, for which we can obtain strong-concavity properties with the quantitative Brunn-Minkowski and isoperimetric inequalities.

Keywords. L_p Minkowski problem, quantitative stability, Wulff shape

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Contents

1. Overview on Minkowski problems

Given a non-trivial finite Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$ of the Euclidean space $\mathbb{R}^n$, Minkowski asked the question under what conditions one can find a convex body whose normal vectors are distributed according to μ. This is now known as Minkowski's problem and has been studied extensively since the beginning of the 20th century. More precisely, let $E \subset \mathbb{R}^n$ be a convex body and let $\nu_E : \partial E \mapsto \mathbb{S}^{n-1}$ denote its Gauss map, when ∂E is a $n-1$-dimensional $\mathscr{C}^1$ manifold $\nu_E(x)$ coincides with the outwards unit normal of ∂E at the point x, then the surface measure of E is given by

$$S_E \overset{\text{def.}}{=} (\nu_E)_\sharp \mathcal{H}^{n-1} \llcorner \partial E, \tag{1.1}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure of $\mathbb{R}^n$. Since $S_{\lambda E} = \lambda^{n-1} S_E$ holds for any $\lambda > 0$, we may assume that our Borel measure μ is a probability distribution; namely, $\mu \in \mathscr{P}(\mathbb{S}^{n-1})$. The *Minkowski problem* can then be phrased as:

Given $\mu \in \mathscr{P}(\mathbb{S}^{n-1})$, is there a convex body E such that $S_E = \mu$?

Extending Minkowski's work [Min03,Min11], this question was answered by Alexandrov [Ale38, Ale96] positively if, and only if, the barycenter in $\mathbb{R}^n$ of the measure μ on $\mathbb{S}^{n-1}$ is the origin and μ is not supported on any hyperplane. He showed that these objects are unique up to translations, and we let E_μ be the unique convex body centered at the origin such that $S_{E_\mu} = \mu$. These are called *Minkowski bodies.*

Whenever $\mathrm{d}\mu = f\mathrm{d}(\mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1})$, the support functions associated with Minkowski bodies solve a corresponding Monge-Ampère equation on $\mathbb{S}^{n-1}$ given

$$\det(\nabla^2_{\mathbb{S}^{n-1}} h + hI) = f, \tag{1.2}$$

where $\nabla_{\mathbb{S}^{n-1}} h$ is the spherical gradient and $\nabla^2_{\mathbb{S}^{n-1}} h$ is the spherical Hessian of h with respect to a moving frame. Regularity of the solution of (1.2) was intensively investigated by Nirenberg [Nir53], Cheng, Yau [CY76] and Pogorelov [Pog64], and the final solution was only provided by Caffarelli [Caf90a,Caf90b] in 1990 (see Figalli [Fig17] and Trudinger, Wang [TW08] for a detailed discussion).

In the present work, we address the question of stability of Minkowski bodies. That is, can we estimate a suitable distance between two Minkowski bodies E_μ, E_ν with a suitable distance between μ, ν? It is well known that Minkowski bodies can be obtained as solutions to an appropriated variational problem, which will be discussed further on, so that the question of stability can be seen as stability of minimizers with respect to a parameter of the energy functional.

In the present work, we prove stability results assuming the form

$$\alpha(E_\mu, E_\nu)^2 \le C \mathrm{d}_{\mathrm{C}}(\mu,\nu)^{1+\frac{1}{n}}, \tag{1.3}$$

for some constant C, and an exponent that is dependent on the dimension n but uniformly bounded from below by its asymptotic $1/2$. Here α denotes the *Fraenkel asymmetry*, used for stability statements by [FMP08] in the case of the isoperimetric inequality and by [FMP10] in the case of the more general anisotropic isoperimetric inequality, defined as

$$\alpha(E,F) \overset{\text{def.}}{=} \inf \left\{ \frac{|E \Delta (x_0 + rF)|}{|E|} : \begin{array}{c} x_0 \in \mathbb{R}^n, \\ r^n|F| = |E|, \ r > 0 \end{array} \right\}. \tag{1.4}$$

It holds that $\alpha(E,F) = \alpha(F,E)$, and $\alpha(E,F) = 0$ if and only if E and F are homothetic, so that α becomes a distance on the equivalence class of homotheties. The *convex-dual distance* d_{C} is defined for two measures μ, ν over $\mathbb{S}^{n-1}$ as

$$\mathrm{d}_{\mathrm{C}}(\mu,\nu) \overset{\text{def.}}{=} \sup_{K \subseteq B_1(0)} \left| \int_{\mathbb{S}^{n-1}} h_K \mathrm{d}(\mu - \nu) \right|, \tag{1.5}$$

where the supremum is taken over all convex bodies included in the unit ball, and h_K corresponds to the support function of K, defined in (2.1). See [AM15] for the basic properties of d_{C}.

Let us briefly discuss the choice of distances in (1.3). Since the support functions h_K are 1-Lipschitz for $K \subset B_1$, whenever μ, ν are probability measures it follows directly that $\mathrm{d}_{\mathrm{C}}(\mu,\nu) \le W_1(\mu,\nu)$, the 1-Wasserstein distance defined via the theory of optimal transport, see [San15, ABS21] for its definition and basic properties. Therefore, a direct corollary is a stability result formulated w.r.t. W_1. Since the Wasserstein distances, on compact ambient spaces, metrize weak convergence of measures, estimate (1.3) provides a way of approximating general convex bodies with polygons, without any regularity or curvature assumption.

Before commenting on the left-hand side of (1.3), let us discuss why we should expect inequality (1.3) to hold. In finite-dimensional variational problems, and more generally in Hilbert spaces, quantitative stability of minimizers is typically a consequence of favorable coercivity properties of the functional being optimized [BS13]. This mechanism is often captured by *Łojasiewicz inequalities* [Loj65], which relate the energy gap to a distance to the set of minimizers. A function f satisfies a Łojasiewicz inequality if it holds for all x that

$$\mathrm{dist}(x, \operatorname{argmin} f)^\theta \le C(f(x) - \inf f) \tag{1.6}$$

for some $\theta > 0$ and $C > 0$.[1] In this sense, the usage of the Fraenkel asymetry in (1.3) reflects naturally these inequalities in the form of a distance to the set o maximizers of the functional $\mathscr{F}_\mu$ used in your proof, see Lemma 3.1 and also Remark 3.3 following it. Such inequalities

[1] For sufficiently smooth energies f, it is equivalent to having an inequality of the form $(f(x) - \inf f)^\theta \le C\|\nabla f(x)\|$.

reflect an underlying curvature of the energy landscape and play a central role in stability and convergence results for associated gradient flows. Indeed, if f is α-strongly convex this identity is true with $\theta = 2$ and $C = \alpha^{-1}$. Extensions of this philosophy to non-linear spaces, such as the Wasserstein space of probability measures, have been developed in recent years, where functional inequalities are reinterpreted as generalized Łojasiewicz inequalities, see for instance [BB18].

The situation for the Minkowski problem is fundamentally different. The natural topologies on the space of convex bodies, such as Hausdorff convergence or L^1-convergence are essentially flat, so instead of leveraging curvature, stability must arise from the rigidity of convex geometry. In this case, convexity/concavity stems naturally from the classical *Brunn-Minkowski inequality*, the reader is referred to [Gar02, Sch14, BFR25] for comprehensive surveys on the many applications of the Brunn-Minkowski theory. It can be stated as follows: given two convex bodies $E, F \subset \mathbb{R}^n$, for all $t \in (0,1)$, we have that

$$|tE + (1-t)F|^{\frac{1}{n}} \geq t|E|^{\frac{1}{n}} + (1-t)|F|^{\frac{1}{n}}, \tag{1.7}$$

where equality holds if and only if E and F are equal up a translation and a dilation.

In this context, we propose a variational problem whose optimizers are translations and dilations of Minkowski bodies associated with a prescribed measure μ, that act as a perturbation parameter. For this energy the quantitative stability of the Brunn-Minkowski inequality, see for instance [FMP09, FJ17, FvHT24], assumes the role of a Łojasiewicz inequality and allows us to deduce stability for Minkowski bodies. This explains the exponent 2 on the Fraenkel asymmetry term as the sharp exponent for the quantitative stability versions of both the Brunn-Minkowski and the Wulff isoperimetric inequalities, see for instance [FMP08, FMP10, FJ17]. The exponent on the right-hand side $1 + \frac{1}{n}$ is obtained by combining a gradient-concavity principle with an older quantitative version of isoperimetry due to Diskant [Dis72].

Previous results in the literature have used the quantitative versions of these inequalities [Dis72], yielding therefore a control of the Hausdorff distance between convex bodies and presenting the worst exponent $1/n$ in the right-hand side. For instance, Hug and Schneider [HS02] obtain stability results conditioned to having bounds on the inner and circumradii of Minkowski bodies; and their results are used in the task of reconstructing convex bodies from random measurements of their normals by Abdallah and Mérigot [AM15].

The conditioning on geometric bounds in [HS02] seems inevitable if we do not make quantitative the assumption of Alexandrov's theorem, that μ is not concentrated on a hyperplane, which is reflected in the constant C from (1.3). Notice that if such an inequality was true for any pair of measures μ, ν, we could take $\mu = \frac{1}{2}\delta_\theta + \frac{1}{2}\delta_{-\theta}$, for some $\theta \in \mathbb{S}^{n-1}$, a sequence $(\mu_k)_{k\in\mathbb{N}}$ not supported in a hyperplane and converging to μ in the Wasserstein distance. As μ_k is a Cauchy sequence for W_1, if (1.3) holds for all measures in $\mathscr{P}(\mathbb{S}^{n-1})$ so will be the associated sequence of Minkowski bodies E_k, which would allow to construct a Minkowski body for μ. This is clearly absurd, since Alexandrov's condition of μ not being supported on a hyperplane is necessary and sufficient for the existence of a Minkowski body.

Therefore, to obtain a result such as (1.3), we need to make quantitative the assumption of μ not being concentrated on a hyperplane. This idea has been exploited in a recent work of the second and third authors [MR25] concerning the quantitative stability of moment measures through the following functional

$$\Theta(\mu) \stackrel{\text{def.}}{=} \inf_{\theta\in\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\theta \cdot x| \mathrm{d}\mu(x). \tag{1.8}$$

If follows directly from the definition that $\Theta(\mu) > 0$ if and only if μ is not supported on a hyperplane, and hence it can be used as quantitative measure of how this condition is satisfied. In addition, if E_μ is the centered Minkowski body associated to μ, we have for all $\theta \in \mathbb{S}^{n-1}$ that

$$\int_{\mathbb{S}^{n-1}} |\theta \cdot y| \mathrm{d}\mu(y) = \int_{\partial E_\mu} |\theta \cdot \nu_{E_\mu}(x)| \mathrm{d}\mathcal{H}^{n-1}(x) = \mathcal{H}^{n-1}(\mathrm{proj}_{\theta^\perp} E_\mu), \tag{1.9}$$

where the last equality is known as *Cauchy's projection formula.* As a result, the quantity $\Theta(\mu)$ gives a lower bound on the size of the projections of E_μ and hence on the size of

E_μ itself. See Figure 1 for an illustration of the role of this functional in the construction proposed in the previous paragraph. Hence, the constant C in (1.3) will be uniform in the class of measures $\mu \in \mathscr{P}(\mathbb{S}^{n-1})$ such that $\Theta(\mu) \geq \vartheta$, and $C = C_{n,\vartheta}$. As a result, the stability of Minkowski bodies degenerates as μ and ν approach measures supported on hyperplanes. The rigorous statement of our quantitative stability result is as follows.

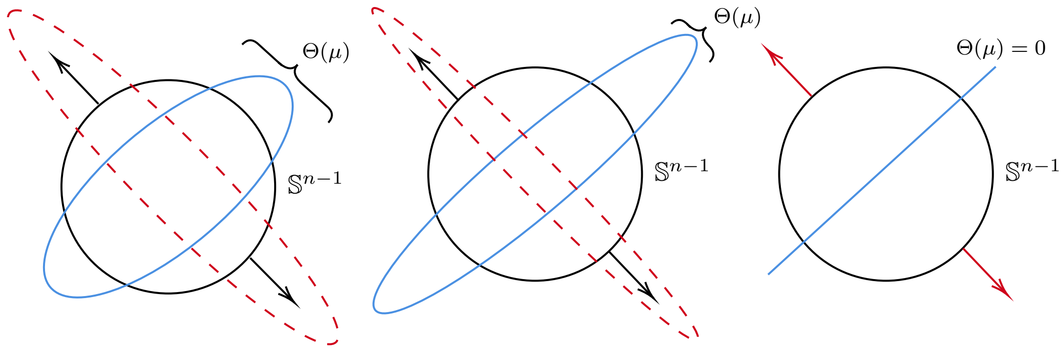

FIGURE 1. Family of ellipsoids collapsing to a line as Θ goes to 0.

Theorem 1.1. *Let $\mu, \nu \in \mathscr{P}(\mathbb{S}^{n-1})$ be two measures such that $\Theta(\mu), \Theta(\nu) \geq \vartheta > 0$, and let E_μ, E_ν be the associated Minkowski bodies centered at the origin. Then there is a constant $C_{\vartheta,n} > 0$ depending only on ϑ and the dimension n such that*

$$\alpha(E_\mu, E_\nu)^2 \leq C_{\vartheta,n} \mathrm{d_C}(\mu,\nu)^{1+\frac{1}{n}}. \tag{1.10}$$

It turns out that our techniques are also applicable to the L_p-Minkowski problems for all $1 \leq p \neq n$. Given a convex body E, Lutwak [Lut93, Lut96] defines the L_p curvature measure as

$$S_{E,p} \overset{\text{def.}}{=} h_E^{1-p} S_E, \tag{1.11}$$

and one can ask the same question as before, under which conditions on $\mu \in \mathscr{P}(\mathbb{S}^{n-1})$ can we find a convex body $E_{\mu,p}$ whose L_p surface measure coincides with μ? In the regime $1 \leq p \neq n$, this problem is well understood, see for instance [CY76, HLYZ05, Bör, HYZ25]. On the other hand, whenever $\mathrm{d}\mu = f\mathrm{d}(\mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1})$ the corresponding Monge-Ampère equation becomes

$$\det(\nabla^2 h + hI) = h^{p-1} f.$$

In this case, the L_p surface measure is not translation invariant, as translating a convex body changes its support function. Therefore, this L_p Minkowski body is unique whenever it exists, instead of unique up to translations, and we no longer need to assume that the measure μ is centered at the origin. In fact, we have existence and uniqueness whenever μ is not concentrated on any half space, that is

$$\mu(H_\theta^+) > 0, \text{ for all } \theta \in \mathbb{S}^{n-1} \text{ where } H_\theta^+ \overset{\text{def.}}{=} \{x \in \mathbb{R}^n : x \cdot \theta > 0\}. \tag{1.12}$$

In a sense, this is the correct dispersion assumption for the $p = 1$ case a well, but with the further assumption that μ is centered, it becomes equivalent to assuming $\Theta(\mu) > 0$. Without the centering assumption on μ in the L_p case, the natural functional to make (1.12) quantitative is

$$\Theta_+(\mu) \overset{\text{def.}}{=} \inf_{\theta \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} (\theta \cdot x)_+ \mathrm{d}\mu(x). \tag{1.13}$$

Since we always have $\Theta(\mu) \geq \Theta_+(\mu)$, we deduce a general control of the geometry of a convex body E by means of the quantity $\Theta(S_{E,p})$. Whereas, for the isoperimetric/Brunn-Minkowski rigidity, we exploit the fact that this variant is closely related to the L_p Brunn-Minkowski theory started by Lutwak [Lut93, Lut96] extending earlier results by Firey [Fir62, Fir74], see also the recent monograph covering the state of the art on this topic [BFR25]. In

Section 4.1, we leverage the quantitative stability for the Brunn-Minkowski and isoperimetric inequalities in the $p = 1$ case to obtain L_p counterparts of such inequalities. This us all the elements necessary to generalize the proof of Theorem 1.1 to the L_p setting. We obtain the following result.

Theorem 1.2. *Let $\mu, \nu \in \mathscr{P}(\mathbb{S}^{n-1})$ be two measures such that $\Theta_+(\mu), \Theta_+(\nu) \geq \vartheta > 0$, and let $E_{\mu,p}, E_{\nu,p}$ be the associated Minkowski bodies centered at the origin. Then there is a constant $C_{\vartheta,n} > 0$ depending only on ϑ and the dimension n such that*

$$\alpha(E_{\mu,p}, E_{\nu,p})^2 \leq C_{\vartheta,n} \mathrm{d_C}(\mu, \nu)^{1+\frac{1}{n}}. \tag{1.14}$$

Concerning the structure of the paper, Section 2 introduces the main notions used in the paper, as well as the control of the geometry of convex bodies with the functional Θ applied to its surface measures and L_p surface measures. The stability versions Theorem 1.1 of the classical Minkowski problem is proved in Section 3, and Theorem 1.2 of the L_p-Minkowski problem for $1 < p \neq n$ is proved in Section 4.

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2. Preliminaries on convex bodies and Minkowski problems

By a convex body $K \subset \mathbb{R}^n$ we mean a closed and convex subset of $\mathbb{R}^n$ with non-empty interior. We say that a convex body K is centered at a certain point x_0 whenever its barycenter lies that x_0. Simply, we say K is centered when its barycenter is the origin, that is if $\int_K x \mathrm{d}x = 0$. Convex bodies are uniquely determined by their support function, given by

$$h_K(x) \overset{\text{def.}}{=} \sup_{y \in K} x \cdot y, \text{ for } x \in \mathbb{S}^{n-1}, \tag{2.1}$$

and then extended uniquely to a 1-homogenous function. It can be easily checked that $K = \{x \in \mathbb{R}^n : x \cdot y \leq h_K(y) \text{ for all } y \in \mathbb{S}^{n-1}\}$, so it encodes many of its geometric properties.

A central object to the present work and to the Brunn-Minkowski theory of convex bodies is the surface area measure. Given a convex body K, we can define its surface measure as the pushforward of the $(n-1)$-dimensional Hausdorff measure on ∂K through the Gauss map $\nu_K : \partial K \mapsto \mathbb{S}^{n-1}$, that is

$$S_K \overset{\text{def.}}{=} (\nu_K)_\sharp \mathcal{H}^{n-1} \llcorner \partial K.$$

Equivalently, for a given Borel set $\omega \subset \mathbb{S}^{n-1}$ it can be written as

$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega)).$$

At any point $x \in \partial K$ such that the normal vector is unique, it coincides with $\nu_K(x)$, so that a surface area measure S_K can be viewed as the distribution of normal vectors associated with the convex body K.

These measures naturally arise as the first variation of the volume functional, or the mixed volumes of convex bodies, see for instance [BFR25]. In particular, if K and L are convex bodies, then the following identity holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} |K + tL| = \int_{\mathbb{S}^{n-1}} h_L(x) \mathrm{d}S_K(x). \tag{2.2}$$

and we define the mixed volume of K and L as

$$V_1(K, L) \overset{\text{def.}}{=} \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(x) \mathrm{d}S_K(x). \tag{2.3}$$

Choosing $K = L$, we obtain that $V(K,K) = |K|$, and it follows from the Brunn-Minkowski inequality (1.7) that

$$V_1(K,L) \geq |K|^{\frac{n-1}{n}}|L|^{\frac{1}{n}}. \tag{2.4}$$

This identity is known by many names, such as the *Wulff isoperimetric inequality*, or the *anisotropic isoperimetric inequality*, and it can be seen as a generalization of the classical isoperimetric inequality, which corresponds to the case $L = B_1(0)$, where $B_1(0)$ is the unit ball centered at the origin.

These identities are fundamental tools to control the geometry of convex bodies, which is done by introducing some quantitative measures of their size. For instance, defining the *inradius* and *circumradius* of a centered convex body K as

$$r_K \stackrel{\text{def.}}{=} \sup\{r > 0 : B_r(0) \subset K\}, \ R_K \stackrel{\text{def.}}{=} \inf\{r > 0 : K \subset B_r(0)\}, \tag{2.5}$$

if follows, whenever $0 \in \operatorname{int} K$, that $0 < r_K \leq h_K \leq R_K$.

A fundamental tool to control the these quantities is the John ellipsoid, which is the unique ellipsoid of maximal volume contained in K. It can be shown that if E is the John ellipsoid of K, then $E \subset K \subset nE$. Letting $a_1 \leq \cdots \leq a_n$ be the half-axes of E, ordered in ascending order so that a_1 is the smallest half-axis, and a_n the largest one, we have that

$$\frac{r_K}{n} \leq a_1 \leq r_K \text{ and } a_n \leq R_K \leq n a_n.$$

The rest of this section is dedicated to obtaining upper bounds on R_K and lower bounds on r_K based on estimates of the form $\Theta(\mu) \geq \vartheta$, where μ is a geometric measure associated with a convex body K, for instance the usual surface measure S_K discussed above, or the L_p-surface measure, relevant to the L_p-Minkowski problem, for $p > 1$, see Section 2.2.

2.1. Inner and circumradii estimates with $\Theta(S_K)$.

As discussed in the introduction, the functional defined over the set of Radon measures on the sphere

$$\mathscr{M}(\mathbb{S}^{n-1}) \ni \mu \mapsto \Theta(\mu) \stackrel{\text{def.}}{=} \inf_{\theta \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\theta \cdot x| \mathrm{d}\mu(x)$$

can be used to obtain quantitative estimates on the geometry of a convex body K. Indeed, thanks to the Cauchy projection formula, we have that given $\theta \in \mathbb{S}^{n-1}$, let $\operatorname{proj}_{\theta^\perp}$ denote the projection operator onto the hyperplane orthogonal to θ, then it holds the following lower bound on the projection area of K onto $\theta^\perp$:

$$\mathcal{H}^{n-1}(\operatorname{proj}_{\theta^\perp}(K)) = \int_{\partial K} |\theta \cdot \nu_K| \mathrm{d}\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} |\theta \cdot x| \mathrm{d}S_K(x) \geq \Theta(S_K). \tag{2.6}$$

This can be used to bound the John ellipsoid of K and hence to obtain estimates on the inner and circumradii of K in terms of $\Theta(S_K)$ and $S(K)$.

Lemma 2.1. *For a convex body K in $\mathbb{R}^n$ there exist dimensional constants $C_n, c_n > 0$ such that*

$$c_n \frac{\Theta(S_K)}{S(K)^{\frac{n-2}{n-1}}} \leq r_K \leq R_K \leq C_n \frac{S(K)}{\Theta(S_K)^{\frac{n-2}{n-1}}},$$

where $c_n = 1/2n^{n^{(n-2)/(n-1)}}$ and $C_n = \frac{n}{2} n^{n^{(n-2)/(n-1)}}$.

Proof. For simplicity of notation let $r = r_K$ and $R = R_K$ denote the inner and circumradii of K. Since translations of the convex body do not change the surface measure, we may assume that the John ellipsoid E of K is centered at the origin. Let $a_1 \leq \cdots \leq a_n$ denote the length of the half axes of E. Since $E \subset K \subset nE$, we have

$$(r/n) \leq a_1 \leq \cdots \leq a_n \leq R \quad \text{and} \quad a_1 \leq r.$$

Choosing the direction θ_n associated with the largest half-axis a_n, from the inclusion the inclusion $K \subseteq nE$ we have that $\operatorname{proj}_{\theta_n^\perp}(K) \subset Q_n \stackrel{\text{def.}}{=} n([-a_1, a_1] \times \cdots \times [-a_{n-1}, a_{n-1}])$. Hence, it follows from the monotonicity of the perimeter that

$$\begin{aligned}\Theta(S_K) &\leq \mathcal{H}^{n-1}(\operatorname{proj}_{\theta_n^\perp}(K)) \leq \mathcal{H}^{n-1}(Q_n) \\ &= (2n)^{n-1} a_1 \cdots a_{n-1} \leq (2n)^{n-1} r R^{n-2}.\end{aligned}$$

On the other hand, choosing the direction θ_1 associated with the smallest half-axis a_1, the inclusion $E \subseteq K$ gives that $Q_1 \overset{\text{def.}}{=} n\left([-a_2, a_2] \times \cdots \times [-a_n, a_n]\right)$. So recalling that $r/n \le a_i$ for all $i = 2, \ldots, n$, and $R/n \le an$, it follows again from the monotonicity of the perimeter that

$$\left(\frac{2}{n}\right)^{n-1} r^{n-2} R \le 2^{n-1} a_2 \cdots a_n = \mathcal{H}^{n-1}(Q_1) \le \mathcal{H}^{n-1}\left(\mathrm{proj}_{\theta_1^\perp}(K)\right) \le S(K).$$

Dividing these identities, using that $r \le a_1$, and $R \le a_n/n$, we obtain that

$$1 \le \frac{R}{r} \le \frac{n a_n}{a_1} = \frac{n a_n \cdots a_2}{a_1 \cdots a_{n-1}} \le n^n \frac{S(K)}{\Theta(S_K)}.$$

Plugging this estimate back into the previous inequalities, we obtain the conclusion. □

2.2. **On the L_p-Minkowski problem for $p > 1$.** Given a convex body K in $\mathbb{R}^n$ with $o \in K$, we can also define the L_p-*surface measures* as

$$\mathrm{d}S_{K,p} = h_K^{1-p} \mathrm{d}S_K. \tag{2.7}$$

Letting $\partial' K$ denote the set of "regular" points $x \in \partial K$ where there exists a unique exterior unit normal $\nu_K(x) \in \mathbb{S}^{n-1}$, we have that for each $x \in \partial' K$ and $y \in K$

$$h_K(x) = x \cdot \nu_K(x) \ge y \cdot \nu_K(x). \tag{2.8}$$

It also holds that $\mathcal{H}^{n-1}(\partial K \backslash \partial' K) = 0$, and as a result, if $\omega \subset \mathbb{S}^{n-1}$ is Borel, the L_p-surface measure of ω can be written as

$$S_{K,p}(\omega) = \int_{\nu_K^{-1}(\omega)} \left(x \cdot \nu_K(x)\right)^{1-p} d\mathcal{H}^{n-1}(x). \tag{2.9}$$

The basic tool to deal with L_p surface measures is Firey's L_p Brunn-Minkowski theory [Fir62]. It is strongly based on the equivalence between a convex body K and its support function h_K. In analogy with the identity $h_{K+L} = h_K + h_L$ from the $p = 1$ case, given two convex bodies K, L containing the origin, and two positive numbers $t, s > 0$, one can define their L_p sum $t \cdot K +_p s \cdot L$ as the convex body whose support function is given by

$$h_{t\cdot K +_p s \cdot L} \overset{\text{def.}}{=} \left(t h_K^p + s h_L^p\right)^{1/p}. \tag{2.10}$$

Notice that, $(K, L, t, s) \mapsto t \cdot K +_p s \cdot L$ is operation depending on the specific choice of each variable, we cannot replace $t \cdot K$ by the dilated body tK in the usual sense. Defining the support function $h_{t\cdot K +_p s\cdot L}$ as above is equivalent to defining the L_p Minkowski sum as

$$t \cdot K +_p s \cdot L \overset{\text{def.}}{=} \left\{x \in \mathbb{R}^n : \langle x, u\rangle \le \left(t h_K^p(u) + s h_L^p(u)\right)^{1/p}, \text{ for all } u \in \mathbb{S}^{n-1}\right\} \tag{2.11}$$

Leveraging the classical Brunn-Minkowski inequality, in the case $p = 1$, it can be shown that a suitable power of the volume functional is concave with respect to the L_p convex combination. Namely, for any pair of convex bodies K, L containing the origin, and $t, s \ge 0$, it holds that

$$|t \cdot K +_p s \cdot L|^{p/n} \ge t|K|^{p/n} + s|L|^{p/n}. \tag{2.12}$$

If before, the first variation of the volume functional, with variations generated by the usual Minkowski sum, gave rise to the usual surface measure S_K, performing L_p variations the first variation of the volume functional gives rise to the L_p surface measure $S_{K,p}$. More precisely, for any pair of convex bodies K, L containing the origin, and any $t > 0$, it holds that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} |K +_p t \cdot L| = \int_{\mathbb{S}^{n-1}} h_L^p(x) \mathrm{d}S_{K,p}(x). \tag{2.13}$$

Mimicking the definition of the mixed volume, we can define the L_p-mixed volume of K and L as

$$V_p(K, L) \overset{\text{def.}}{=} \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L^p(x) \mathrm{d}S_{K,p}(x), \tag{2.14}$$

and as a consequence of the L_p Brunn-Minkowski inequality, the L_p-Wulff isoperimetric inequality also holds. Namely, for any pair of convex bodies K, L containing the origin, it holds that

$$V_p(K,L) \geq |K|^{\frac{n-p}{n}} |L|^{\frac{p}{n}}. \tag{2.15}$$

The L_p Minkowski problem for $p > 1$ then becomes the question of existence and uniqueness of a convex body K such that $S_{K,p} = \mu$, for a given measure $\mu \in \mathscr{P}(\mathbb{S}^{n-1})$. Differently from the case $p = 1$, the L_p surface measure is not translation invariant. So we cannot assume $o \in \operatorname{int} K$, and we have to consider the case $o \in \partial K$. As a result, for the condition $d\mu = h_K^{1-p}\, dS_K$ to make sense, we must check that if $o \in \partial K$, then $S_K(\{h_K = 0\}) = 0$.

Readily, the following conditions are equivalent for a convex body $K \subset \mathbb{R}^n$ with $o \in \partial K$:

- $S_K(\{h_K = 0\}) = 0$;
- $S_K(N_K(o) \cap \mathbb{S}^{n-1}) = 0$, where $N_K(o) = \{y \in \mathbb{R}^n : y \cdot x \leq 0 \text{ for } x \in K\}$ is the exterior normal cone at $o \in \partial K$;
- $\mathcal{H}^{n-1}(\{x \in \partial' K : \nu_K(x) \cdot x = 0\}) = 0$.

Remark. The condition $S_K(\{h_K = 0\}) = 0$ is a consequence if we write the Monge-Ampère equation in the form

$$\mathrm{d}S_K = h_K^{p-1}\, \mathrm{d}\mu,$$

or in the traditional form

$$\det(\nabla^2 h + hI) = h^{p-1} f.$$

In particular, the solution of the L_p-Minkowski problem for $p > 1$ reads as follows.

Theorem 2.2. *Let μ be a finite Borel measure on $\mathbb{S}^{n-1}$. Then $\mu = S_{K,p}$; namely, the L_p surface area measure of a convex body K with $o \in K$ and $S_K(\{h_K = 0\}) = 0$ if and only if μ is not concentrated on any closed hemisphere.*

The hypothesis that μ is not concentrated on any closed hemisphere is equivalent saying that

$$\Theta_+(\mu) \overset{\text{def.}}{=} \inf_{\theta \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} (\theta \cdot v)_+ \, dv > 0. \tag{2.16}$$

This restriction is strictly stronger than asking for a lower bound on the functional $\Theta(\mu)$, defined in (1.8). Indeed, if $\Theta_+(\mu) \geq \vartheta$, then $\Theta(\mu) \geq 2\vartheta$, since we can write

$$\int_{\mathbb{S}^{n-1}} |\theta \cdot v| \mathrm{d}\mu(v) = \int_{\mathbb{S}^{n-1}} (\theta \cdot v)_+ \mathrm{d}\mu(v) + \int_{\mathbb{S}^{n-1}} ((-\theta) \cdot v)_+ \mathrm{d}\mu(v) \geq 2\vartheta.$$

As a result, the class of measures that admit a solution to the L_p-Minkowski problem for $p > 1$ is strictly smaller than for the $p = 1$ case. On the other hand, obtaining bounds on the inner and circumradii of a convex body K in terms of $\Theta(S_{K,p})$ is stronger than obtaining such bounds in terms of $\Theta_+(S_{K,p})$.

The following Lemma about the functionals Θ and Θ_+ will be useful in the sequel.

Lemma 2.3. *Let μ be a finite Borel measure on $\mathbb{S}^{n-1}$. Define the sets*

$$\Omega_{\theta,\delta} = \{v \in \mathbb{S}^{n-1} : |\theta \cdot v| > \delta\}.$$

Then, if $\Theta(\mu) > 0$, it holds that $\mu(\Omega_{\theta,\delta}) > \delta$ for all $\theta \in \mathbb{S}^{n-1}$, where δ is given by $\delta = \min\{\frac{\Theta(\mu)}{2\mu(\mathbb{S}^{n-1})}, \frac{\Theta(\mu)}{2}\}$.

Proof. Fix $\theta \in \mathbb{S}^{n-1}$ and consider $\delta = \min\{\frac{\Theta(\mu)}{2\mu(\mathbb{S}^{n-1})}, \frac{\Theta(\mu)}{2}\}$. We can write

$$0 < \Theta(\mu) \leq \int_{\mathbb{S}^{n-1}} |\theta \cdot v| \mathrm{d}\mu(v) \leq \delta\mu(\mathbb{S}^{n-1}) + \mu(\Omega_{\theta,\delta}).$$

As a result, we have $\mu(\Omega_{\theta,\delta}) \geq \Theta(\mu) - \delta\mu(\mathbb{S}^{n-1}) \geq \frac{\Theta(\mu)}{2} \geq \delta$, and the results follows. Naturally, the same conclusion holds if we replace $\Theta(\mu)$ by $\Theta_+(\mu)$, and $\Omega_{\theta,\delta}$ by $\Omega^+_{\theta,\delta} = \{v \in \mathbb{S}^{n-1} : (\theta \cdot v)_+ > \delta\}$. □

Next, we will show that if $\Theta(S_{K,p})$ is bounded from below, then the inner and circumradii of K are bounded from below and above, respectively. We start with the case $1 < p < n$.

Lemma 2.4. *Given $1 < p < n$, consider a convex body K such that $o \in K$, $S_K(\{h_K = 0\}) = 0$, $\Theta(S_{K,p}) > 0$, and normalized so that*

$$S_{K,p}(\mathbb{S}^{n-1}) = 1.$$

Then it holds that

$$c_{n,p}\Theta(S_{K,p})^{\underline{\beta}} \le r_K \le R_K \le C_{n,p}\Theta(S_{K,p})^{-\overline{\beta}},$$

for some constants $c_{n,p}, C_{n,p} > 0$ and exponents $\underline{\beta}, \overline{\beta} > 0$ depending only on n, p. In particular, we have $\overline{\beta} = \frac{p}{p-1}$ and $\underline{\beta} = \frac{(n-1)np}{(p-1)(n-p)}$.

Proof. We start with the following application of Hölder's inequality: note that

$$\int_{\mathbb{S}^{n-1}} |\langle x, \theta\rangle| \,\mathrm{d}S_K(x) = \int_{\mathbb{S}^{n-1}} |\langle x, \theta\rangle| \, h_K(x)^{-\frac{p-1}{p}} \, h_K(x)^{\frac{p-1}{p}} \,\mathrm{d}S_K(x).$$

Using Hölder's inequality, this yields that the left-hand side above is bounded by

$$\left(\int_{\mathbb{S}^{n-1}} |\langle x, \theta\rangle| \, h_K(x)^{1-p} \,\mathrm{d}S_K(x)\right)^{\frac{1}{p}} \left(\int_{\mathbb{S}^{n-1}} |\langle x, \theta\rangle| \, h_K(x) \,\mathrm{d}S_K(x)\right)^{\frac{p-1}{p}}.$$

$$= \left(\int_{\mathbb{S}^{n-1}} |\langle x, \theta\rangle| \,\mathrm{d}S_{K,p}(x)\right)^{\frac{1}{p}} \left(\int_{\mathbb{S}^{n-1}} |\langle x, \theta\rangle| \, h_K(x) \,\mathrm{d}S_K(x)\right)^{\frac{p-1}{p}}$$

$$\le S_{K,p}(\mathbb{S}^{n-1})^{1/p} \left(\int_{\mathbb{S}^{n-1}} |\langle x, \theta\rangle| \, h_K(x) \,\mathrm{d}S_K(x)\right)^{\frac{p-1}{p}}.$$

Integrating both sides in $\theta \in \mathbb{S}^{n-1}$, and using that $S_{K,p}(\mathbb{S}^{n-1}) = 1$, we have

$$\int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{S}^{n-1}} |\langle x, \theta\rangle| \,\mathrm{d}S_K(x)\right) \mathrm{d}\mathcal{H}^{n-1}(\theta)$$

$$\le S_{K,p}(\mathbb{S}^{n-1})^{1/p} \int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{S}^{n-1}} |\langle x, \theta\rangle| \, h_K(x) \,\mathrm{d}S_K(x)\right)^{\frac{p-1}{p}} \mathrm{d}\mathcal{H}^{n-1}(\theta).$$

Now, by using Hölder's inequality on the right-hand side and Fubini's theorem on the left-hand side, we get that

$$\sigma_{n-1}\, S(K) \le S_{K,p}(\mathbb{S}^{n-1})^{1/p}\sigma_{n-1}^{1/p} \left(\int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{S}^{n-1}} |\langle x, \theta\rangle| \, h_K(x) \,\mathrm{d}S_K(x)\right)^{\frac{p}{p-1}\cdot\frac{p-1}{p}} \mathrm{d}\mathcal{H}^{n-1}(\theta)\right)^{\frac{p-1}{p}}$$

(2.17)

$$= S_{K,p}(\mathbb{S}^{n-1})^{1/p}\sigma_{n-1} \left(\int_{\mathbb{S}^{n-1}} h_K \,\mathrm{d}S_K\right)^{\frac{p-1}{p}} = \sigma_{n-1} n^{\frac{p-1}{p}}\, V(K)^{\frac{p-1}{p}},$$

where $\sigma_{n-1} = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$ is the surface area of the unit sphere in $\mathbb{R}^n$.

Next, we bound the perimeter with a power of the volume. Using the L_p-Wulff inequality (4.6) applied to the convex body K and with F being the unit ball, we obtain

$$1 = S_{K,p}(\mathbb{S}^{n-1}) = \int_{\mathbb{S}^{n-1}} h_K^{1-p} \,\mathrm{d}S_K \ge n\, \omega_n^{\frac{p}{n}}\, V(K)^{\frac{n-p}{n}}.$$

Since we have supposed that $p \in (1, n)$, it follows at once that the volume of K is bounded by an absolute constant times the L_p perimeter of K. Plugging this back into (2.17), it follows that the perimeter of K is bounded by an absolute constant

$$S(K) \le c_{n,p}, \text{ where } c_{n,p} \text{ depends only on } n \text{ and } p. \tag{2.18}$$

Now, we note that since $\Theta(S_{K,p}) > 0$, taking $\delta = \min\left\{\frac{1}{2}, \frac{1}{2}\Theta(S_{K,p})\right\}$ it holds that

$$\inf_{u\in\mathbb{S}^{n-1}} \int_{\{x\in\mathbb{S}^{n-1}\,:\, |\langle x,u\rangle|\ge\delta\}} |\langle x, u\rangle| \,\mathrm{d}S_{K,p}(x) > \delta.$$

Since $S_{K,p}(\mathbb{S}^{n-1}) = 1$, a simple inspection gives that $\delta = \frac{1}{2}\Theta(S_{K,p})$.

Letting R_K and r_K denote the circumradius and inradius of K, respectively, consider $u_0 \in \mathbb{S}^{n-1}$ to be a direction on the sphere such that $R_K \cdot u_0 \in \overline{K}$. Hence, if $x \in \mathbb{S}^{n-1}$ belongs to the set where $|\langle x, u_0\rangle| \geq \delta$, it follows by the previous discussion that $h_K(x) \geq \delta \cdot R_K$. Hence,

$$\begin{aligned} \delta &\leq \int_{\{x\in\mathbb{S}^{n-1}\colon\, |\langle x,u_0\rangle|\geq\delta\}} |\langle x, u_0\rangle| \,\mathrm{d}S_{K,p}(x) \\ &= \int_{\{x\in\mathbb{S}^{n-1}\colon\, |\langle x,u_0\rangle|\geq\delta\}} |\langle x, u_0\rangle|\, h_K^{1-p} \,\mathrm{d}S_K(x) \\ &\leq \delta^{1-p} R_K^{1-p} S_K(\{x \in \mathbb{S}^{n-1}\colon |\langle x, u_0\rangle| \geq \delta\}). \end{aligned} \tag{2.19}$$

We now estimate the perimeter above in two different ways, both starting out by estimating

$$S_K(\{x \in \mathbb{S}^{n-1}\colon |\langle x, u_0\rangle| \geq \delta\}) \leq S(K).$$

Indeed, from (2.18) we know that $S(K)$ is bounded by a constant depending only on n and p. Secondly, we have that, as K is contained in a ball of radius R_K,

$$S(K) \leq \sigma_{n-1} \cdot R_K^{n-1}.$$

Since $1 < p < n$, the first of those inequalities entails in an *upper bound* on R_K. More precisely, combing (2.19) with (2.18), we get the upper bound on R_K

$$R_K \leq C_{n,p}\Theta(S_{K,p})^{-\overline{\beta}}, \tag{2.20}$$

where it can be checked that the exponent is given by $\overline{\beta} = \frac{p}{p-1}$.

On the other hand, the easy estimate $S(K) \leq \sigma_{n-1}R_K^{n-1}$ combined with (2.19) gives the lower bound on R_K

$$R_K \geq \bar{c}_{n,p}\Theta(S_{K,p})^{\frac{p}{n-p}}, \tag{2.21}$$

for some constant $\bar{c}_{n,p}$ depending only on n and p. Combining (2.21),(2.19), and (2.17) we get a lower bound on the volume of K depending only on n, p and $\Theta(S_{K,p})$, namely there is another constant $\tilde{c}_{n,p}$ such that

$$V(K) \geq \tilde{c}_{n,p}\, \Theta(S_{K,p})^{\frac{n-1}{n-p}\frac{p^2}{p-1}}, \tag{2.22}$$

Finally, in order to finish the proof, we note that, by taking the John ellipsoid E of K, with $a_1 \leq \cdots \leq a_n$ being its half axes, then

$$2^n \cdot r_K \cdot R_K^{n-1} \geq 2^n \cdot a_1 \cdot \cdots \cdot a_n \geq |K| \geq \tilde{c}_n.$$

This, together with the upper bound we have on R_K, implies that r_K admits a lower bound of the form

$$r_K \geq c_{n,p}\Theta(S_{K,p})^{\underline{\beta}},$$

where $c_{n,p}$ is a constant depending only on the dimension, p and the exponent $\underline{\beta}$ can be checked to be $\underline{\beta} = \frac{(n-1)np}{(p-1)(n-p)}$. This finishes the proof. □

Some observations concerning the previous Lemma 2.4 in comparison with Lemma 2.1 for the case $p = 1$ are in order.

Remark 2.5. Note that all the constants can be made explicit and similar bounds can be deduced for general convex bodies, without the normalization $S_{K,p}(\mathbb{S}^{n-1}) = 1$. This simplification is very natural for our quantitative stability result and was made in order to keep the presentation simpler, while still conveying the dependency on $\Theta(S_{K,p})$, which is the quantity that controls the geometry of K.

Remark 2.6. Notice that the exponents of $\Theta(S_{K,p})$ do not converge to the exponent of $\Theta(S_K)$ in Lemma 2.1 as $p \to 1$. This is not surprising, and is an artifact of the L_p surface measure not being translation invariant. It is not clear however if the exponents are optimal.

In the case $p > n$, the above arguments are greatly simplified, since the corresponding L_p-Wulff inequality gives directly a lower bound on the volume of any convex body such that the L_p-perimeter is normalized to 1. For our purposes in quantitative stability the lower bound on the inner radius is only used to bound the volume from below, so this step can be by-passed completely. But since we expect this result to be useful in broader situations, we include a strategy to obtain quantitative lower bounds on the inner radius as well.

Lemma 2.7. *Given $p > n$, consider a convex body K such that $o \in K$, $S_K(\{h_K = 0\}) = 0$, $\Theta(S_{K,p}) > 0$, and normalized so that*

$$S_{K,p}(\mathbb{S}^{n-1}) = 1.$$

Then its volume is bounded from below by a universal constant $|K| \geq V_{n,p} \stackrel{\text{def.}}{=} n^{\frac{n}{p-n}} \omega_n^{\frac{p}{p-n}}$, and we have the following bounds

$$c_{n,p}\Theta(S_{K,p})^{p\frac{n-1}{p-n}} \leq r_K \leq R_K \leq C_{n,p}\Theta(S_{K,p})^{-\frac{p}{p-n}}.$$

Proof. First we notice that an easy application of the L_p-Wulff inequality (2.15) gives that

$$1 = S_{K,p}(\mathbb{S}^{n-1}) \geq n\omega_n^{\frac{p}{n}} |K|^{\frac{n-p}{n}},$$

so as $p > n$, we have that the volume of K is bounded from below by the constant stated above.

Now, we note that since $\Theta(S_{K,p}) > 0$, taking $\delta = \min\left\{\frac{1}{2}, \frac{1}{2}\Theta(S_{K,p})\right\}$ it holds that

$$\inf_{u\in\mathbb{S}^{n-1}} \int_{\{x\in\mathbb{S}^{n-1}\colon |\langle x,u\rangle|\geq\delta\}} |\langle x, u\rangle| \,\mathrm{d}S_{K,p}(x) > \delta.$$

Since $S_{K,p}(\mathbb{S}^{n-1}) = 1$, a simple inspection gives that $\delta = \frac{1}{2}\Theta(S_{K,p})$.

To bound the circumradius from above, we can use a similar argument as in the case $1 < p < n$. Since $\Theta(S_{K,p}) > 0$, defining once again the quantity $\delta = \frac{1}{2}\Theta(S_{K,p})$, and taking a direction u_0 such that $R_K u_0 \in \partial K$, we have

$$\begin{aligned}
\delta &\leq \int_{\{x\in\mathbb{S}^{n-1}\colon |\langle x,u_0\rangle|\geq\delta\}} |\langle x, u_0\rangle| \,\mathrm{d}S_{K,p}(x) \\
&= \int_{\{x\in\mathbb{S}^{n-1}\colon |\langle x,u_0\rangle|\geq\delta\}} |\langle x, u_0\rangle| \, h_K^{1-p} \,\mathrm{d}S_K(x) \\
&\leq \delta^{1-p} R_K^{1-p} S_K(\{x \in \mathbb{S}^{n-1}\colon |\langle x, u_0\rangle| \geq \delta\}) \\
&\leq \sigma_{n-1}\delta^{1-p} R_K^{n-p}.
\end{aligned}$$

We hence obtain the desired upper bound on R_K

$$R_K \leq C_{n,p}\Theta(S_{K,p})^{-\frac{p}{p-n}}, \tag{2.23}$$

with an explicit constant $C_{n,p} \stackrel{\text{def.}}{=} \sigma_{n-1}^{\frac{1}{p-n}}$.

To finish the proof, we use a simplified argument to the one in the case $1 < p < n$. By taking the John ellipsoid E of K, with $a_1 \leq \cdots \leq a_n$ being its half axes, then

$$c_n \cdot r_K \cdot R_K^{n-1} \geq c_n \cdot a_1 \cdot \cdots \cdot a_n \geq |K| \geq \tilde{c}_n.$$

This, together with the upper bound we have on R_K, implies that r_K admits a lower bound of the form

$$r_K \geq c_{n,p}\Theta(S_{K,p})^{p\frac{n-1}{p-n}},$$

and the result follows. □

3. Quantitative stability for $p = 1$

Now we are ready to prove our main stability result for Minkowski's problem. Our proof relies on first leveraging the rigidity conferred by anisotropic isoperimetry, and then using it to reinforce a stability principle based on a variational formulation of Minkowski problem, for which the qualitative stability theory of the Brunn-Minkowski inequality plays a major role. Our major reference for these tools are [Dis72, FMP10], whose results we review briefly in the sequel.

Any convex body F containing the origin induces an anisotropy as the quantity

$$\|v\|_F \stackrel{\text{def.}}{=} \sup\{v \cdot x : x \in F\}, \quad v \in \mathbb{S}^{n-1}.$$

Of course, this is just the geometric measure theory notation for the support function, $\|\cdot\|_F = h_F$. Whereas the anisotropic perimeter of a set E

$$P_F(E) \stackrel{\text{def.}}{=} \int_{\partial E} \|\nu_E(x)\|_F \,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_{\mathbb{S}^{n-1}} h_F \mathrm{d}S_E = nV_1(E, F) \tag{3.1}$$

and can also be equivalently written by means of the first mixed volume.

One of the first approach to reinforce this inequality with quantitative version is due to Diskant [Dis72], where a control of the Hausdorff distance between two convex bodies can be obtained assuming bounds on the geometry of the convex bodies involved. Recalling that the Hausdorff distance of convex bodies can be written as the L_∞ norm between the corresponding support functions

$$d_H(E,F) = \|h_E - h_F\|_\infty ,$$

it is shown in [Dis72] that, given two convex bodies E, F such that $r \le r_E, r_F$ and $R_E, R_F \le R$, there is a constant C depending on r, R, n such that

$$d_H(E,F)^n \le C\left(V_1(E,F) - |E|^{\frac{n-1}{n}}|F|^{\frac{1}{n}}\right). \tag{3.2}$$

The optimal version of this inequality was then proposed in [FMP10], where the dependence on the geometry is removed by considering the following notion of discrepancy for Wulff's inequality

$$0 \le \delta_{\mathrm{ISO}}(E,F) \overset{\text{def.}}{=} \frac{P_F(E)}{n|F|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}} - 1 = \frac{V_1(E,F)}{|F|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}} - 1, \tag{3.3}$$

which controls then the stronger Fraenkel asymmetry between E and F

$$\alpha(E,F) \overset{\text{def.}}{=} \inf\left\{\frac{|E\Delta(x_0+rF)|}{|E|} \,:\, x_0\in\mathbb{R}^n,\ r^n|F| = |E|\right\}. \tag{3.4}$$

In other words, there is a dimensional constant C_n, for which the anisotropic isoperimetric inequality can be strengthened as

$$\alpha(E,F)^2 \le C_n \delta_{\mathrm{ISO}}(E,F). \tag{3.5}$$

Our second approach to the stability of Minkowski bodies employs the sharp quantitative stability of the Brunn-Minkowski inequality. As we will only need this inequality with the convexity parameter given by $1/2$, we define the deficit of the Brunn-Minkowski inequality for two convex sets E, F as

$$\delta_{\mathrm{BM}}(E,F) \overset{\text{def.}}{=} \frac{\left|\frac{1}{2}E + \frac{1}{2}F\right|^{\frac{1}{n}}}{\frac{1}{2}|E|^{\frac{1}{n}} + \frac{1}{2}|F|^{\frac{1}{n}}} - 1. \tag{3.6}$$

Differently from the isoperimetric inequality, in this case the roles of E and F are completely symmetric, so it is only natural to compensate with ratio between the volumes of the sets

$$\sigma_{E,F} \overset{\text{def.}}{=} \max\left\{\frac{|E|}{|F|}, \frac{|F|}{|E|}\right\},$$

then there exists a constant C_n depending only on the dimension for which the following stronger version holds

$$\alpha(E,F)^2 \le C_n\, \sigma_{E,F}^{\frac{1}{n}}\, \delta_{\mathrm{BM}}(E,F). \tag{3.7}$$

3.1. **Proof of Theorem 1.1.** As mentioned above, our second approach is to exploit variational formulation that describes Minkowski bodies to derive stability estimates. A classical variational problem describing them is the following

$$\lambda_\mu \overset{\text{def.}}{=} \inf_{K \text{ centered}} F_\mu(K) \overset{\text{def.}}{=} \frac{\displaystyle\int_{\mathbb{S}^{n-1}} h_K \mathrm{d}\mu}{|K|^{\frac{1}{n}}}. \tag{3.8}$$

Notice that the energy F_μ is invariant with respect to dilations. Therefore, we can show with Wulff's inequality that the only minimizers of F_μ are dilations of Minkowski's bodies associated with μ.

Lemma 3.1. *The following assertions hold:*

(1) *Let $\mu \in \mathscr{P}(\mathbb{S}^{n-1})$ such that $\Theta(\mu) > 0$, then the quantity $K \mapsto F_\mu(K)$ is minimized by any dilation of the Minkowski body E_μ associated with μ. As a result we have*

$$\lambda_\mu = n|E_\mu|^{\frac{n-1}{n}},$$

where E_μ is the unique centered body such that $S_{E_\mu} = \mu$.

(2) *For μ, ν such that $\Theta(\mu), \Theta(\nu) > 0$ it holds that*

$$|\lambda_\mu - \lambda_\nu| \leq C_{\mu,\nu}\, \mathrm{d_C}(\mu, \nu),$$

where W_1 corresponds to the 1-Wasserstein distance and $C_{\mu,\nu} = \max\left\{\frac{R(E_\mu)}{|E_\mu|^{\frac{1}{n}}}, \frac{R(E_\nu)}{|E_\nu|^{\frac{1}{n}}}\right\}$.

Proof. To prove item (1), first notice that for any centered convex body L, by the anisotropic isoperimetric inequality, with anisotropy given by h_L, we have

$$\int_{\mathbb{S}^{n-1}} h_L \mathrm{d}\mu = \int_{\mathbb{S}^{n-1}} h_L \mathrm{d}S_{E_\mu} = \int_{\partial E_\mu} h_L(\nu_{E_\mu}) \mathrm{d}\mathcal{H}^{n-1} \geq n|L|^{\frac{1}{n}}|E_\mu|^{\frac{n-1}{n}}.$$

It then follows that $F_\mu(L) \geq n|E_\mu|^{\frac{n-1}{n}}$ for all centered convex bodies L. Choosing $L = E_\mu$, from the identity $\int h_E \mathrm{d}S_E = n|E|$ valid for any centered convex body, we see that E_μ attains the lower bound and so must be optimal for λ_μ. Since this energy is invariant with respect to dilations, this conclusion holds for any $K = \lambda E_\mu$ and $\lambda > 0$.

To prove item (2), consider two measures μ, ν and use $F_\mu(E_\nu)$ to bound λ_μ from above. Since the support function h_{E_ν} is Lipchitz continuous with constant given by the circumradius of E_ν we have

$$\lambda_\mu - \lambda_\nu \leq \frac{1}{|E_\nu|^{\frac{1}{n}}} \int_{\mathbb{S}^{n-1}} h_{E_\nu} \mathrm{d}(\mu - \nu) \leq \frac{R(E_\nu)}{|E_\nu|^{\frac{1}{n}}}\, \mathrm{d_C}(\mu, \nu).$$

Changing the roles of μ and ν, the result follows. □

Although fundamental, it is less clear how to exploit the stability of Brunn-Minkowski's inequality and the variational problem (3.8) to obtain the desired stability estimates. A variant of a problem recently proposed by Carlier in [Car04], consists of exploring the equivalence between a convex body and its support function. Indeed, obtainning this function is clear, but given $\varphi \in \mathscr{C}_b(\mathbb{S}^{d-1})$, we can construct its associated convex body as

$$E_\varphi \stackrel{\text{def.}}{=} \left\{x \in \mathbb{R}^n : x \cdot \nu \leq \varphi(\nu)\ \nu \in \mathbb{S}^{n-1}\right\}, \text{ so that } h_{E_\varphi} = \varphi. \tag{3.9}$$

We could then consider an energy of the form $\varphi \mapsto |E_\varphi|^{\frac{1}{n}} - \int_{\mathbb{S}^{n-1}} \varphi \mathrm{d}\mu$, which has the advantage of being convave due to the Brunn-Minkowski inequality, and we can hope to extract information from maximizers from (3.7).

The only issue is that this energy scales linearly with dilation, so we multiply it by a suitable constant to fix a natural scale of the energy. This is done by considering

$$\sup_{\varphi \in \mathscr{C}_b(\mathbb{S}^{n-1})} \mathscr{F}_\mu(\varphi) \stackrel{\text{def.}}{=} \lambda_\mu |E_\varphi|^{\frac{1}{n}} - \int_{\mathbb{S}^{n-1}} \varphi \mathrm{d}\mu. \tag{3.10}$$

With this constant, the scale is fixed for the maximization of $\mathscr{F}_\mu$, and it selects precisely the Minkowski bodies, modulo translations, associated with μ.

Lemma 3.2. *A function φ is maximizer of $\mathscr{F}_\mu$ if and only if E_φ is homotethic to the Minkowski body E_μ associated with μ.*

Proof. Let φ_μ be the support function associated with E_μ the centered Minkowski body of μ. By the equality case of the anisotropic isoperimetric inequality from above, for any support function φ that is not equivalent to φ_μ, for a translation and a dilation, we have

$$\mathscr{F}_\mu(\varphi) = \lambda_\mu |E_\varphi|^{\frac{1}{n}} - \int_{\mathbb{S}^{n-1}} \varphi \mathrm{d}\mu < \frac{\int_{\mathbb{S}^{n-1}} \varphi \mathrm{d}\mu}{|E_\varphi|^{\frac{1}{n}}} |E_\varphi|^{\frac{1}{n}} - \int_{\mathbb{S}^{n-1}} \varphi \mathrm{d}\mu = 0.$$

In addition, we have $\mathscr{F}_\mu(\varphi_\mu) = 0$, from the fact that the centered Minkowski body E_μ is optimal for λ_μ. □

Remark 3.3. From the previous Lemma we can indeed appreciate the thesis of Theorem 1.1 as a Łojasiewicz inequality in the sense of (1.6). Indeed, the equivalence class of homotheties that is hard encoded into the Fraenkel asymmetry is precisely the set of maximizers of the functionals $\mathscr{F}_\mu$.

The energy (3.10) is convenient since the first variation of the volume functional can be explicitly computed with the surface measure as discussed above (2.2). Indeed, for any $\varphi, v \in \mathscr{C}_b(\mathbb{S}^{n-1})$ and defining $\varphi_t \stackrel{\text{def.}}{=} (1-t)\varphi + tv$ we have

$$\lim_{t\to 0} \frac{|E_{\varphi_t}| - |E_\varphi|}{t} = \int_{\partial E_\varphi} v(\nu_{E_\varphi}) \mathrm{d}\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} v \mathrm{d}S_{E_\varphi}, \tag{3.11}$$

As a result, the functional $\varphi \mapsto \mathscr{F}_\mu(\varphi)$ is concave and its first variation evaluated at a function φ is the Radon measure given by

$$\nabla \mathscr{F}_\mu[\varphi] = \lambda_\mu \frac{S_{E_\varphi}}{n|E_\varphi|^{\frac{n-1}{n}}} - \mu. \tag{3.12}$$

As a result, the first order optimality condition for (3.10) become $\nabla\mathscr{F}_\mu[\varphi] \equiv 0$, in the sense of measures, which is equivalent to being a Minkowski body, as we have already proven.

At this point we have all the elements to obtain the desired stability result.

Proof of Theorem 1.1. Given $\mu, \nu \in \mathscr{P}(\mathbb{S}^{n-1})$ such that $\Theta(\mu), \Theta(\nu) \geq \vartheta > 0$, the strict positivity of the functional Θ implies the existence of centered Minkowski bodies E_μ, E_ν. Let φ_μ, φ_ν be their corresponding support functions. We first obtain a weaker stability result by bounding the isoperimetric deficit. As a consequence of Lemma 2.1, there are universal constants $C_{n,\vartheta}, c_{n,\vartheta} > 0$ depending only on n and ϑ such that

$$c_{n,\vartheta} \leq r(E_\mu) \leq R(E_\mu) \leq C_{n,\vartheta},$$

and similarly for E_ν. As a result, up to replacing the constants above, it follows that

$$c_{n,\vartheta} \leq |E_\mu|, |E_\nu|, \quad \mathrm{Lip}(\varphi_\mu), \mathrm{Lip}(\varphi_\nu) \leq C_{n,\vartheta}.$$

Without loss of generality we can assume that $|E_\mu| \geq |E_\nu|$, which implies with the anisotropic isoperimetric inequality that

$$V_1(E_\mu, E_\nu) - |E_\nu| \geq V_1(E_\mu, E_\nu) - |E_\nu|^{\frac{1}{n}} |E_\mu|^{\frac{n-1}{n}}.$$

Next we recall the following identities for the mixed volumes

$$V_1(E_\mu, E_\nu) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \varphi_\nu \mathrm{d}S_{E_\mu}, \quad |E_\nu| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \varphi_\nu \mathrm{d}S_{E_\nu},$$

which combined with Diskant's quantitative isoperimetric inequality (3.2), and using that $S_{E_\mu} = \mu, S_{E_\nu} = \nu$, it follows from the definition of the dual-convex distance $\mathrm{d_C}$ (1.5) that

$$\|\varphi_\mu - \varphi_\nu\|_\infty \leq C\, \mathrm{d_C}(\mu,\nu)^{\frac{1}{n}}, \tag{3.13}$$

where C is a constant depending only on ϑ and n.

We can obtain a stronger version by employing the Figalli-Maggi-Pratelli quantitative version of the isoperimetric inequality (3.4), obtaining an estimate of the form $\alpha(E_\mu, E_\nu) \leq C_{n,\vartheta}\, \mathrm{d_C}(\mu,\nu)^{1/2}$. This is also achievable with the variational argument that follows, this time using the Brunn-Minkowski inequality, but the latter combined with (3.13) yields a better exponent.

Now, recall that from Lemma 3.1 we know that both support functions φ_μ, φ_ν maximize the functionals $\mathscr{F}_\mu$ and $\mathscr{F}_\nu$, respectively. Hence, we define $\varphi_{1/2} \stackrel{\text{def.}}{=} \frac{1}{2}(\varphi_\mu + \varphi_\nu)$, and consider the following quantity

$$\begin{aligned} 0 \leq \delta &\stackrel{\text{def.}}{=} \mathscr{F}_\mu(\varphi_{1/2}) - \frac{1}{2}\left(\mathscr{F}_\mu(\varphi_\mu) + \mathscr{F}_\mu(\varphi_\nu)\right) \\ &= \left|\frac{1}{2}E_\mu + \frac{1}{2}E_\nu\right|^{\frac{1}{n}} - \frac{1}{2}\left(|E_\mu|^{\frac{1}{n}} + |E_\nu|^{\frac{1}{n}}\right) \\ &= \frac{1}{2}\left(|E_\mu|^{\frac{1}{n}} + |E_\nu|^{\frac{1}{n}}\right)\delta_{\mathrm{BM}}(E_\mu, E_\nu). \end{aligned} \tag{3.14}$$

The quantity δ above is non-negative thanks to the concavity of $\varphi \mapsto \mathscr{F}_\mu(\varphi)$. So, using the fact that φ_μ is a maximizer and the aforementioned concavity, it follows that

$$\begin{aligned}
\delta &\le \frac{1}{2}\left(\mathscr{F}_\mu(\varphi_\mu) - \mathscr{F}_\mu(\varphi_\mu)\right) \le \frac{1}{2}\left\langle \nabla \mathscr{F}_\mu(\varphi_\nu), \varphi_\mu - \varphi_\nu \right\rangle \\
&= \frac{1}{2}\left\langle \frac{\lambda_\mu}{\lambda_\nu} S_{E_\nu} - \mu, \varphi_\mu - \varphi_\nu \right\rangle = \frac{1}{2\lambda_\nu}\left\langle \lambda_\mu \nu - \lambda_\nu \mu, \varphi_\mu - \varphi_\nu \right\rangle \\
&= \frac{\lambda_\mu}{2\lambda_\nu}\left\langle \nu - \mu, \varphi_\mu - \varphi_\nu \right\rangle + \frac{\lambda_\mu - \lambda_\nu}{2\lambda_\nu}\left\langle \mu, \varphi_\mu - \varphi_\nu \right\rangle \\
&\le \frac{\lambda_\mu}{2\lambda_\nu}\left\|\varphi_\mu - \varphi_\nu\right\|_\infty \int_{\mathbb{S}^{n-1}} 1\mathrm{d}(\nu - \mu) + \frac{\lambda_\nu}{2}\mu(\mathbb{S}^{n-1})\left\|\varphi_\mu - \varphi_\nu\right\|_\infty (\lambda_\nu - \lambda_\mu) \\
&\le C_{n,\vartheta}\, \mathrm{d_C}(\mu,\nu)^{1+\frac{1}{n}},
\end{aligned}$$

where we have used the quantitative stability of the constant $\mu \mapsto \lambda_\mu$ with respect to $\mathrm{d_C}(\mu,\nu)$, proved in Lemma 3.1, combined with the uniform estimates on the inradius and circumradius from 2.1. Combining the above estimate with (4.12) and the quantitative stability of the Brunn-Minkowski inequality (3.7), we conclude that there exists a constant $C_{\vartheta,n}$ such that

$$\alpha(E_\mu, E_\nu)^2 \le C_{n,\vartheta}\, \mathrm{d_C}(\mu,\nu)^{1+\frac{1}{n}},$$

and the result follows. □

4. Quantitative stability for $1 < p \neq n$

In order to adapt the previous arguments for the $p > 1$ case, the central object becomes Firey's L_p Brunn-Minkowski theory. As already discussed in Section 2.2, the L_p theory is strongly based on the equivalence between a convex body E and its support function h_E, we recall that the L_p-sum of convex bodies is defined via a non-linear combination of their respective support functions as

$$h_{t\cdot E +_p s\cdot F} \overset{\text{def.}}{=} (th_E^p + sh_F^p)^{1/p}. \tag{4.1}$$

The convex body that defined the corresponding L_p Minkowski sum is then given by

$$t\cdot E +_p s\cdot F \overset{\text{def.}}{=} \left\{x \in \mathbb{R}^n : \langle x, u\rangle \le (th_E(u) + sh_F(u))^{\frac{1}{n}},\ \text{for all } u \in \mathbb{S}^{n-1}\right\} \tag{4.2}$$

As in the classical $p = 1$ case, a suitable power of the volume functional is concave with respect to the L_p Minkowski sum. Namely, for any pair of convex bodies E, F which contain the origin it holds that

$$|t\cdot E +_p s\cdot F|^{p/n} \ge t|E|^{p/n} + (1-t)|F|^{p/n}. \tag{4.3}$$

This theory has already given us the analogous control to the case $p = 1$ of the geometry of a convex body E, whenever the quantity $\Theta(S_{E,p}) > 0$, is strictly positive. Since this was the major technical result that allowed our uniform control in the proof of Theorem 1.1, this encourages the extension of both arguments – the variational one, and via isoperimetric rigidity – to the L_p case.

To the best of the authors' knowledge, there is no "off-the-shelf" quantitative stability results for the L_p isoperimetric and Brunn-Minkowski inequalities, in analogies to the classical couterparts (3.4) and (3.7). For this reason we dedicate the next paragraph to the derivation of these stronger versions in the L_p case.

4.1. Quantitative versions of L_p Brunn-Minkowski and isoperimetric inequalities.

Firey's L_p Brunn-Minkowski inequalities can be derived from the classical $p = 1$ case, see for instance [BFR25, Thm. 7.6.3] or [Fir62]. On the other hand, as was already remarked in Section 2.2, the L_p isoperimetric inequalities can be derived from the L_p-BM theory. Therefore, in this section we simply use the stronger quantitative versions from the $p = 1$ case in the proof of these results.

Lemma 4.1. *Given $p > 1$, let $E, F \subset \mathbb{R}^n$ be convex bodies containing the origin. Then there is some $\tau \in (0, 1/2]$ such that $\tau \leq t \leq 1-\tau$, and a dimensional constant $\theta_n \overset{\text{def.}}{=} cn^{-4}(\log n)^{-1}$, with $c \in (0,1)$ such that*

$$|t \cdot E +_p (1-t) \cdot F| \geq |E|^t |F|^{1-t} \left(1 + \theta_n \tau \alpha(E,F)^2\right). \tag{4.4}$$

In addition, for all $t, s > 0$ we have

$$|t \cdot E +_p s \cdot F|^{p/n} \geq \left(t|E|^{p/n} + s|F|^{p/n}\right) \left(1 + \frac{\theta_n}{\sigma_{E,F}^{p/n}} \alpha(E,F)^2\right)^{p/n}, \tag{4.5}$$

where $\sigma_{E,F} \overset{\text{def.}}{=} \max\left\{\frac{|E|}{|F|}, \frac{|F|}{|E|}\right\}$.

Proof. The proof is obtained by using the quantitative stability version of the Brunn-Minkowski inequality for convex sets into the proof of the L_p Brunn-Minkowski inequality found for instance in [BFR25].

For the first case, take $t \in (0,1)$, so that by convexity of the function $r \mapsto r^p$ for $p > 1$, we have that

$$h^p_{t \cdot E +_p (1-t) \cdot F} = t h^p_E + (1-t) h^p_F \geq (t h_E + (1-t) h_F)^p.$$

So that

$$tE + (1-t)F \subseteq t \cdot E +_p (1-t) \cdot F.$$

Using the quantitative stability version of the classical Brunn-Minkowski inequality [BFR25, Corollary 8.6.5], this implies that

$$|t \cdot E +_p (1-t) \cdot F| \geq |tE + (1-t)F| \geq |E|^t |F|^{1-t} \left(1 + \theta_n \tau \alpha(E,F)^2\right),$$

where $\tau \in (0, 1/2]$ is a such that $\tau \leq t \leq 1-\tau$, and $\theta_n \overset{\text{def.}}{=} cn^{-4}(\log n)^{-1}$.

Now let us derive the second inequality set $t_0 \overset{\text{def.}}{=} |E|^{\frac{1}{n}}$, $s_0 \overset{\text{def.}}{=} |F|^{\frac{1}{n}}$, as well as the dilations

$$E_0 \overset{\text{def.}}{=} {t_0}^{-1} E, \quad F_0 \overset{\text{def.}}{=} {s_0}^{-1} F,$$

in such a way that $|E_0| = |F_0| = 1$. Therefore, considering

$$\lambda \overset{\text{def.}}{=} \frac{s\, s_0^p}{t\, t_0^p + s\, s_0^p} \in (0,1),$$

and assuming without loss of generality that $\frac{s\ s_0^p}{t\, t_0^p + s\, s_0^p} < \frac{t\, t_0^p}{t\, t_0^p + s\, s_0^p}$, we have that

$$\begin{aligned} h_{t \cdot E +_p s \cdot F} &= (t\, h^p_E + s\, h^p_F)^{1/p} \\ &= (t\, t_0^p + s\, s_0^p)^{1/p} \left((1-\lambda) h^p_{E_0} + \lambda h^p_{F_0}\right)^{1/p} \end{aligned}$$

As a result, using the previous quantitative stability version of the Brunn-Minkowski inequality with the renormalized sets E_0, F_0, we get that

$$\begin{aligned} (t\, t_0^p + s\, s_0^p)^{\frac{n}{p}} |t \cdot E +_p s \cdot F| &= |(1-\lambda) \cdot E_0 +_p \lambda \cdot F_0| \\ &\geq \left(1 + \theta_n \tau \alpha(E_0, F_0)^2\right), \end{aligned}$$

for τ such that

$$\lambda \leq \tau \leq 1 - \lambda.$$

Recalling the definition of t_0 and s_0 above, and dilating E_0, F_0 back to E, F it clearly holds that $\alpha(E_0, F_0) = \alpha(E, F)$ and we get that

$$|t \cdot E +_p s \cdot F|^{p/n} \geq \left(t|E|^{p/n} + s|F|^{p/n}\right) \left(1 + \frac{\theta_n}{\sigma_{E,F}^{p/n}} \alpha(E,F)^2\right)^{p/n}$$

□

This proof is most likely not optimal due to the exponent p/n that seems inevitable with this approach. It is nevertheless sufficient to conclude with the quantitative stability result in the L_p case.

On the other hand, leveraging the quantitative anisotropic isoperimetric inequality, we can sharpen the inequality on the L_p mixed volume

$$V_p(E,F) \geq |F|^{\frac{p}{n}}|E|^{\frac{n-p}{n}}, \text{ equivalent to } 0 \leq \delta_{\mathrm{ISO},p}(E,F) \overset{\text{def.}}{=} \frac{P_{F,p}(E)}{n|F|^{\frac{p}{n}}|E|^{\frac{n-p}{n}}} - 1, \tag{4.6}$$

where we have equality if and only if E and F are dilates.

Lemma 4.2. *Given $p > 1$, there exists a constant $C_n > 0$, depending only on the dimension, such that for any convex bodies E, F we have that*

$$C_n\, \alpha(E,F)^2 \leq \delta_{ISO,p}(E,F). \tag{4.7}$$

Using Diskant's version of the quantitative isoperimetric inequality, given $0 < r < R$, there is a constant C depending on r, R, p and n such that for any pair of convex bodies E, F such that $r \leq r_E, r_F$ and $R_E, R_F \leq R$, it holds that

$$\|h_E - h_F\|_\infty^n \leq C\left(V_p(E,F) - |E|^{\frac{n-p}{n}}|F|^{\frac{p}{n}}\right). \tag{4.8}$$

Proof. Defining the volume measure $V_E \overset{\text{def.}}{=} \frac{1}{n}h_E S_E$, we recall that $|E| = V_E(\mathbb{S}^{n-1})$. As a result, from Jensen's inequality we have that

$$\begin{aligned}
V_p(E,F)^{1/p} &= |E|^{1/p}\left(\int_{\mathbb{S}^{n-1}} \frac{h_F^p}{h_E^p}\frac{\mathrm{d}V_E}{|E|}\right)^{1/p} \geq |E|^{1/p}\int_{\mathbb{S}^{n-1}} \frac{h_F}{h_E}\frac{\mathrm{d}V_E}{|E|}\\
&= |E|^{\frac{1}{p}-1}\frac{1}{n}\int_{\mathbb{S}^{n-1}} h_F \mathrm{d}S_E = |E|^{\frac{1}{p}-1}V_1(E,F)\\
&\geq |F|^{\frac{1}{n}}|E|^{\frac{1}{p}-\frac{1}{n}}\left(1 + C_n\alpha(E,F)^2\right),
\end{aligned}$$

where in the last identity we have used the quantitative anisotropic isoperimetric inequality. This yields that

$$V_p(E,F) \geq |F|^{\frac{p}{n}}|E|^{\frac{n-p}{n}}\left(1 + C_n\alpha(E,F)^2\right)^p.$$

As $p > 1$ and the quantity on the p-th power on the right-hand side is at least 1, we have that

$$C_n\,\alpha(E,F)^2 \leq \delta_{\mathrm{ISO},p}(E,F),$$

and the result follows.

By employing Diskant's version of the isoperimetric inequality (3.2) in the above argument, we obtain (4.8). □

4.2. **Proof of Theorem 1.2.** We can also adapt the variational proof developed in Section 3.1 to prove Theorem 1.2. In this case, using the L_p Brunn-Minkowski theory, the quantity $E \mapsto |E|^{p/d}$ is concave for the L_p Minkowski sum. For the formulation in terms of functions $\varphi \in \mathscr{C}_b(\mathbb{S}^{n-1})$, we have that the quantity

$$\varphi \mapsto |E_\varphi|^{\frac{p}{n}}$$

is concave but for a non-linear type of variation, *i.e.* $\varphi_t \mapsto |E_{\varphi_t}|^{\frac{p}{n}}$ is concave with $\varphi_t \overset{\text{def.}}{=} ((1-t)\varphi_0^p + t\varphi_1^p)^{1/p}$.

As a result, if before the central idea was balancing a suitable power of the volume with a linear functional induced by the measure μ, now we need to consider functionals which are linear w.r.t. the type of variations that make $\varphi \mapsto |E_\varphi|^{\frac{p}{n}}$ concave. So in this framework the corresponding linear functional becomes

$$\varphi \mapsto \int_{\mathbb{S}^{n-1}} \varphi^p \mathrm{d}\mu.$$

Once again, the quantities $t \mapsto |E_{t\varphi}|^{\frac{p}{n}}$ and $t \mapsto \int_{\mathbb{S}^{n-1}} (t\varphi)^p \mathrm{d}\mu$ scale both like t^p. So to maximize the corresponding energy to (3.10), we need to fix the scale with a suitable constant that will be defined as

$$\lambda_{\mu,p} \overset{\text{def.}}{=} \inf_K F_{\mu,p}(K) \overset{\text{def.}}{=} \frac{\displaystyle\int_{\mathbb{S}^{n-1}} h_K{}^p \mathrm{d}\mu}{|K|^{\frac{p}{n}}}, \tag{4.9}$$

where the infimum is taken among all convex bodies K such that $o \in K$ and $S_{K,p}(\{h_K = 0\}) = 0$. So the energy whose maximization yields L_p Minkowski bodies becomes

$$\sup_{\varphi \in \mathscr{C}_b(\mathbb{S}^{n-1})} \mathscr{F}_{\mu,p}(\varphi) \overset{\text{def.}}{=} \lambda_{\mu,p}|E_\varphi|^{p/n} - \int_{\mathbb{S}^{n-1}} \varphi^p \mathrm{d}\mu. \tag{4.10}$$

As in the $p = 1$, with this constant, the correct scale is fixed for the maximization of $\mathscr{F}_{\mu,p}$, and it selects precisely the Minkowski bodies associated with μ.

Lemma 4.3. *For all $1 \le p \neq n$, the following assertions hold:*

(1) Let $\mu \in \mathscr{P}(\mathbb{S}^{n-1})$ such that $\Theta_+(\mu) > 0$, then the quantity $K \mapsto F(K)$ is minimized by any dilation of the L_p Minkowski body $E_{\mu,p}$ associated with μ. As a result we have

$$\lambda_{\mu,p} = n|E_{\mu,p}|^{\frac{n-p}{n}},$$

where $E_{\mu,p}$ is the unique convex body such that $S_{E_{\mu,p},p} = \mu$.

(2) A support function φ maximizes the energy $\mathscr{F}_{\mu,p}$ if, and only if, it is a dilation of the support function $\varphi_{\mu,p}$ associated with the L_p Minkowski body $E_{\mu,p}$ of μ.

(3) For μ, ν such that $\Theta(\mu), \Theta(\nu) > 0$ it holds that

$$|\lambda_{\mu,p} - \lambda_{\nu,p}| \le C_{\mu,\nu}\, \mathrm{d_C}(\mu,\nu),$$

where $C_{\mu,\nu} = \max\left\{\frac{R(E_{\mu,p})}{|E_{\mu,p}|^{p/n}}, \frac{R(E_{\nu,p})}{|E_{\nu,p}|^{p/n}}\right\}$.

The proof of this Lemma is completely analogous to the $p = 1$ case detailed in Section 3.1.

Proof of Thm. 1.2. Consider $\mu, \nu \in \mathscr{P}(\mathbb{S}^{n-1})$ such that $\Theta_+(\mu), \Theta_+(\nu) \ge \vartheta > 0$. From the positivity of the functional Θ_+, Theorem 2.2 asserts the existence of L_p Minkowski bodies $E_{\mu,p}, E_{\nu,p}$ associated with μ, ν. Let $\varphi_{\mu,p}$ and $\varphi_{\nu,p}$ be the corresponding support functions.

It follows from Lemmas 2.4 and 2.7, in the cases $1 < p < n$ and $n < p$ respectively, that there are positive constants $C_{n,p,\vartheta}, c_{n,p,\vartheta} > 0$ depending on n, p, and ϑ such that

$$c_{n,p,\vartheta} \le |E_{\mu,p}|, |E_{\nu,p}|, \quad \mathrm{Lip}(\varphi_{\mu,p}), \mathrm{Lip}(\varphi_{\nu,p}) \le C_{n,p,\vartheta}.$$

With a similar argument from the $p = 1$ case, assuming that $|E_{\mu,p}| \ge |E_{\nu,p}|$, and recalling that

$$V_p(E_{\mu,p}, E_{\nu,p}) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \varphi_{\nu,p}^p \mathrm{d}S_{E_{\mu,p}}, \quad |E_{\nu,p}| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \varphi_{\nu,p}^p \mathrm{d}S_{E_{\nu,p}},$$

and that $S_{E_{\mu,p},p} = \mu, S_{E_{\nu,p},p} = \nu$, we obtain from the Hausdorff stability of the L_p-isoperimetric inequality (4.8) that

$$\|\varphi_{\mu,p} - \varphi_{\nu,p}\|_\infty \le C_{n,p,\vartheta}\, \mathrm{d_C}(\mu,\nu)^{\frac{1}{n}}. \tag{4.11}$$

As in the $p = 1$ case, at this point we can use the strong Fraenkel asymmetry rigidity from Lemma 4.2 to obtain an estimate of the form $\alpha(E_{\mu,p}, E_{\nu,p}) \le C_{n,p,\vartheta}\mathrm{d_C}(\mu,\nu)^{1/2}$, but once again, we can obtain a better exponent with the variational formulation described above.

We know from Lemma 4.3 that $\varphi_{\mu,p}$ and $\varphi_{\nu,p}$ maximize the functionals $\mathscr{F}_{\mu,p}$ and $\mathscr{F}_{\nu,p}$, respectively. As in the $p = 1$ case, we want to exploit the concavity of $\mathscr{F}_{\mu,p}$, but this time this concavity is expressed in terms of the following nonlinear variation

$$\varphi_{1/2} \overset{\text{def.}}{=} \left(\frac{1}{2}\varphi_{\mu,p}^p + \frac{1}{2}\varphi_{\nu,p}^p\right)^{1/p},$$

which also has the advantage of preserving the portion of the energy coming from $\varphi \mapsto \int_{\mathbb{S}^{n-1}} \varphi^p \mathrm{d}\mu$.

From the L_p Brunn-Minkowski inequality, the following quantity is non-negative

$$
\begin{aligned}
0 \le \delta &\overset{\text{def.}}{=} \mathscr{F}_{\mu,p}(\varphi_{1/2}) - \frac{1}{2}\left(\mathscr{F}_{\mu,p}(\varphi_\mu) + \mathscr{F}_{\mu,p}(\varphi_\nu)\right) \\
&= \left|\frac{1}{2}E_{\mu,p} + \frac{1}{2}E_{\nu,p}\right|^{\frac{1}{n}} - \frac{1}{2}\left(|E_{\mu,p}|^{p/n} + |E_{\nu,p}|^{p/n}\right) \\
&\quad + \int_{\mathbb{S}^{n-1}} \underbrace{\left(\varphi_{1/2}^p - \frac{1}{2}\varphi_{\mu,p}^p - \frac{1}{2}\varphi_{\nu,p}^p\right)}_{=0} \mathrm{d}\mu \\
&= \left|\frac{1}{2}E_{\mu,p} + \frac{1}{2}E_{\nu,p}\right|^{p/n} - \frac{1}{2}\left(|E_{\mu,p}|^{p/n} + |E_{\nu,p}|^{p/n}\right).
\end{aligned} \tag{4.12}
$$

Since the volumes $|E_{\mu,p}|, |E_{\nu,p}|$ are uniformly controlled from above and from below, up to using a worse constant in the quantitative L_p Brunn-Minkowski inequality from 4.1 we have the bound

$$
\begin{aligned}
\left|\frac{1}{2}E_{\mu,p} + \frac{1}{2}E_{\nu,p}\right|^{p/n} &\ge \frac{1}{2}\left(|E_{\mu,p}|^{p/n} + |E_{\nu,p}|^{p/n}\right)\left(1 + C'_{n,p,\vartheta}\ \alpha(E_{\mu,p}, E_{\nu,p})^2\right)^{p/n} \\
&\ge \frac{1}{2}\left(|E_{\mu,p}|^{p/n} + |E_{\nu,p}|^{p/n}\right)\left(1 + \frac{p}{n}C'_{n,p,\vartheta}\ \alpha(E_{\mu,p}, E_{\nu,p})^2\right)
\end{aligned} \tag{4.13}
$$

So that

$$
\alpha(E_{\mu,p}, E_{\nu,p})^2 \le C_{n,p,\vartheta}\ \delta. \tag{4.14}
$$

As in the $p = 1$ case, to control the quantity δ above we use the concavity of the function $t \mapsto F(t) \overset{\text{def.}}{=} \mathscr{F}_{\mu,p}(\varphi_t)$, for $\varphi_t \overset{\text{def.}}{=} \left((1-t)\varphi_{\nu,p}^p + t\varphi_{\mu,p}^p\right)^{1/p}$, along with the optimality of $\mathscr{F}_{\mu,p}(\varphi_{1/2}) \le \mathscr{F}_{\mu,p}(\varphi_{\mu,p})$. Using the stability in Hausdorff distance from (4.11), along with the analogous arguments from the $p = 1$ case, we have that

$$
\begin{aligned}
\delta &= \mathscr{F}_{\mu,p}(\varphi_{1/2}) - \frac{1}{2}\left(\mathscr{F}_{\mu,p}(\varphi_{\mu,p}) + \mathscr{F}_{\mu,p}(\varphi_{\mu,p})\right) \le \frac{1}{2}\left(\mathscr{F}_{\mu,p}(\varphi_{\mu,p}) - \mathscr{F}_{\mu,p}(\varphi_{\mu,p})\right) \\
&\le \frac{1}{2}F'(0) = \frac{1}{2}\left\langle \nabla_p \mathscr{F}_{\mu,p}(\varphi_{\nu,p}), \varphi_{\mu,p} - \varphi_{\nu,p}\right\rangle \\
&= \frac{1}{2}\left\langle \frac{\lambda_{\mu,p}}{\lambda_{\nu,p}} S_{E_{\nu,p}} - \mu, \varphi_{\mu,p} - \varphi_{\nu,p}\right\rangle = \frac{1}{2\lambda_{\nu,p}}\left\langle \lambda_{\mu,p}\nu - \lambda_{\nu,p}\mu, \varphi_{\mu,p} - \varphi_{\nu,p}\right\rangle \\
&\le C_{n,p,\vartheta}\mathrm{d}_{\mathrm{C}}(\mu,\nu)^{1+\frac{1}{n}},
\end{aligned}
$$

where we have used the quantitative stability of the constant $\mu \mapsto \lambda_{\mu,p}$ with respect to $\mathrm{d}_{\mathrm{C}}(\mu,\nu)$, from Lemma 4.3. Combining this estimate with (4.14) the result follows. ☐

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