On the effective sharp stability of the Faber–Krahn inequality

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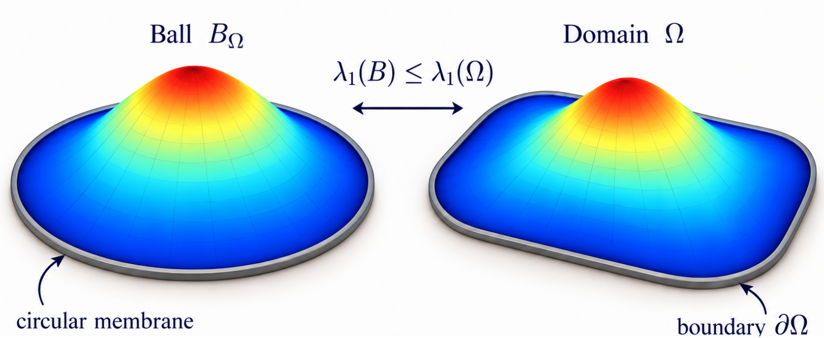
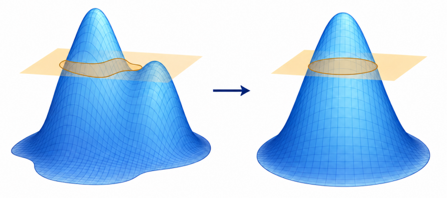
The Faber–Krahn inequality

$$\lambda_1(\Omega) = \inf_{u \in W_0^{1,2}(\Omega)} \left\{ \int_\Omega |\nabla u|^2 \, dx : \quad \|u\|_{L^2(\Omega)} = 1 \right\}$$

At fixed volume, balls minimize the first Dirichlet eigenvalue.



The classical Pólya–Szegő proof



For $u \geq 0$, set $m_u(t) \stackrel{\text{def.}}{=} |\{u > t\}|$. Define

$$u^\circ(s) \stackrel{\text{def.}}{=} \inf\{t \geq 0 : m_u(t) \leq s\}, \qquad 0 < s < |\Omega|,$$
$$u^*(x) \stackrel{\text{def.}}{=} u^\circ(\omega_n |x|^n), \qquad x \in B_\Omega.$$

Here B_Ω is centered at 0, and $\|u^*\|_2 = \|u\|_2$.

Pólya–Szegő: rearrangement does not increase the Dirichlet energy.

$$\int_{B_\Omega} |\nabla u^*|^2 \, \mathrm{d}x \leq \int_\Omega |\nabla u|^2 \, \mathrm{d}x.$$

For a first eigenfunction u, normalized by $\|u\|_2 = 1$,

$$\lambda_1(B_\Omega) \leq \int_{B_\Omega} |\nabla u^*|^2 \leq \int_\Omega |\nabla u|^2 = \lambda_1(\Omega).$$

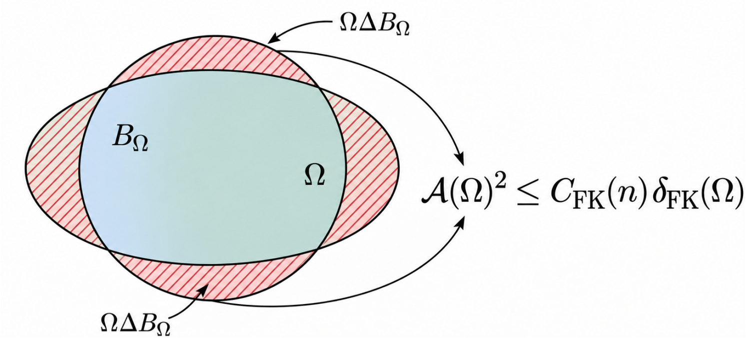
Quantitative stability: how close is the drum to a ball?

Distance to the family of balls

$$\mathcal{A}(\Omega) = \inf_{z\in\mathbb{R}^n} \frac{|\Omega \,\Delta\, B_\Omega(z)|}{|\Omega|}.$$

Scale-invariant deficit

$$\delta_{\mathrm{FK}}(\Omega) = |\Omega|^{2/n}\lambda_1(\Omega) - |B|^{2/n}\lambda_1(B).$$



Sharp stability: a bit of history

Earlier stability used stronger geometric distances

- **Diskant (1972):** for convex bodies, the isoperimetric deficit controls a Hausdorff-type deviation from a ball.[1]
- **Fuglede (1989):** write a nearly spherical boundary as a radial graph and control its deformation by the perimeter deficit; convexity gives a global result.[2]
- **Hansen–Nadirashvili (1994) and Melas (1992):** early quantitative Faber–Krahn estimates use the inradius or nested-ball distances, under convexity or planar assumptions.[3]

For arbitrary open sets, symmetric difference is the robust distance.

[1] Diskant (1972), "Deviation of convex bodies and isoperimetric difference".
[2] Fuglede (1989), "Stability in the isoperimetric problem".
[3] Hansen and Nadirashvili (1994), "Isoperimetric inequalities in potential theory"; Melas (1992), "The stability of some eigenvalue estimates".

Sharp stability: a bit of history

Geometric foundation: quantitative isoperimetry

$$\mathcal{A}(E)^2 \leq C(n)\,\frac{\mathrm{Per}(E) - \mathrm{Per}(B_E)}{\mathrm{Per}(B_E)}, \qquad |B_E| = |E|.$$

- **Fusco–Maggi–Pratelli (2008):** the sharp quadratic inequality, proved by symmetrization.[1]
- **Figalli–Maggi–Pratelli (2010):** a mass-transport proof and the anisotropic inequality.[2]
- **Cicalese–Leonardi (2012):** the selection principle, based on penalization and regularity.[3]

[1] Fusco, Maggi, and Pratelli (2008), "The sharp quantitative isoperimetric inequality".
[2] Figalli, Maggi, and Pratelli (2010), "A mass transportation approach to quantitative isoperimetry".
[3] Cicalese and Leonardi (2012), "A selection principle for quantitative isoperimetry".

Sharp stability: a bit of history

Result	**Estimate**	**Constant**
Fusco–Maggi–Pratelli (2009)[1]	$\mathcal{A}^4 \leq C(n)\delta_{\mathrm{FK}}$	computable
Brasco–De Philippis (2017)[2]	$\mathcal{A}^3 \leq C(n)\delta_{\mathrm{FK}}$	computable
Brasco–De Philippis–Velichkov (2015)[3]	$\mathcal{A}^2 \leq C(n)\delta_{\mathrm{FK}}$	non-explicit

- The power 2 is optimal.
- The sharp proof uses the **Cicalese–Leonardi selection principle** and contradiction.

[1] Fusco, Maggi, and Pratelli (2009), "Stability estimates for Faber–Krahn inequalities".
[2] Brasco and De Philippis (2017), "Spectral inequalities in quantitative form".
[3] Brasco, De Philippis, and Velichkov (2015), "Faber–Krahn inequalities in sharp quantitative form".

Effective sharp stability

Brasco–De Philippis, Open Problem 2: obtain a computable constant with the sharp exponent.[1]

Guerra–M.–Ramos (2026)

For every $n \geq 2$ there is a computable $C_{\mathrm{FK}}(n) > 0$ such that every open $\Omega \subset \mathbb{R}^n$, $0 < |\Omega| < \infty$, satisfies

$$\boxed{\mathcal{A}(\Omega)^2 \leq C_{\mathrm{FK}}(n)\, \delta_{\mathrm{FK}}(\Omega).}$$

The constants and all smallness thresholds can be traced through the proof.

Route: torsion level sets, their speed of motion, and strong quantitative isoperimetry.[2]

[1] Brasco and De Philippis (2017), "Spectral inequalities in quantitative form", Open Problem 2.
[2] Fusco and Julin (2014), "A strong form of the quantitative isoperimetric inequality"; Hensel and Laux (2026), "Quantitative isoperimetry: a calibration argument".

Saint–Venant stability transfers to Faber–Krahn

Saint–Venant

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad T(\Omega) = \int_\Omega u \,\mathrm{d}x.$$

Balls maximize torsion at fixed volume:

$$T(\Omega) \leq T(B_\Omega), \qquad |B_\Omega| = |\Omega|.$$

For $|\Omega| = \omega_n \stackrel{\text{def.}}{=} |B_1|$, set

$$\delta_{\mathrm{SV}}(\Omega) \stackrel{\text{def.}}{=} T(B_1) - T(\Omega).$$

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Kohler–Jobin[a]

$$\lambda_1(\Omega)\, T(\Omega)^{\frac{2}{n+2}} \ge \lambda_1(B_1)\, T(B_1)^{\frac{2}{n+2}}.$$

Consequently,

$$\delta_{\mathrm{SV}}(\Omega) \le C(n)\delta_{\mathrm{FK}}(\Omega),$$

where $C(n)$ is computable.

[a] Kohler-Jobin (1978), "Une méthode de comparaison isopérimétrique"; Brasco (2014), "On torsional rigidity and principal frequencies".

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$$\mathcal{A}(\Omega)^2 \le C(n) \delta_{\mathrm{SV}}(\Omega) \quad \Longrightarrow \quad \boxed{\mathcal{A}(\Omega)^2 \le C(n) \delta_{\mathrm{FK}}(\Omega)}$$

A level-set proof: the required notation

For the torsion function $u = u_\Omega$, set

$$E_t = \{u > t\}, \qquad m(t) = |E_t|, \qquad c_n = n^2 \omega_n^{2/n}.$$

For almost every t, coarea and the equation $-\Delta u = 1$ give

$$\alpha(t) \overset{\text{def.}}{=} \int_{\partial^* E_t} \frac{\mathrm{d}\mathcal{H}^{n-1}}{|\nabla u|} = -m'(t), \qquad \beta(t) \overset{\text{def.}}{=} \int_{\partial^* E_t} |\nabla u| \, \mathrm{d}\mathcal{H}^{n-1} = m(t).$$

Two nonnegative defects

$$D_H(t) \overset{\text{def.}}{=} \alpha(t)\beta(t) - \mathrm{Per}(E_t)^2 \geq 0 \qquad \text{(Cauchy–Schwarz)},$$

$$D_I(t) \overset{\text{def.}}{=} \mathrm{Per}(E_t)^2 - c_n m(t)^{2-2/n} \geq 0 \qquad \text{(isoperimetry)}.$$

$\partial^* E_t$: reduced boundary; $\mathrm{Per}(E_t) = \mathcal{H}^{n-1}(\partial^* E_t)$.

The torsional deficit separates two mechanisms

Theorem (Exact torsional deficit identity)

$$\delta_{\mathrm{SV}}(\Omega) = \int_0^{\|u\|_\infty} \frac{D_H(t) + D_I(t)}{c_n m(t)^{1-2/n}}\,\mathrm{d}t.$$

Proof: the rearrangement profile. For $0 < s < \omega_n$, let $u^\circ(s) = \inf\{t : m(t) \le s\}$. Then

$$m(u^\circ(s)) = s, \qquad (u^\circ)'(s) = -\frac{1}{\alpha(u^\circ(s))} \quad \text{a.e.}$$

Set

$$I(s) = \int_0^s u^\circ(\sigma)\,\mathrm{d}\sigma, \qquad G(s) = I(s) + \frac{s^{1+2/n}}{2(n+2)\omega_n^{2/n}}.$$

Thus

$$G'(s) = u^\circ(s) + \frac{s^{2/n}}{2n\omega_n^{2/n}}, \qquad G''(s) = -\frac{1}{\alpha(u^\circ(s))} + \frac{s^{2/n-1}}{c_n}.$$

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A simple integration, using the change of variables $s = m(t)$, gives

$$\int_0^{\omega_n} sG''(s)\,\mathrm{d}s = \int_0^{\omega_n} \left(\frac{s^{2/n}}{c_n} - \frac{s}{\alpha(u^\circ(s))} \right) \mathrm{d}s = \int_0^{\|u\|_\infty} \left(\frac{m(t)^{2/n}}{c_n} - \frac{m(t)}{\alpha\big(\underbrace{u^\circ(m(t))}_{=t}\big)} \right) \alpha(t)\,\mathrm{d}t$$

$$= \int_0^{\|u\|_\infty} \left(\frac{m(t)\alpha(t)}{c_n m(t)^{1-2/n}} - m(t) \right) \mathrm{d}t$$

$$= \int_0^{\|u\|_\infty} \left(\frac{\alpha(t)\beta(t) - \mathrm{Per}(E_t)^2}{c_n m(t)^{1-2/n}} + \frac{\mathrm{Per}(E_t)^2 - c_n m(t)^{2-2/n}}{c_n m(t)^{1-2/n}} \right) \mathrm{d}t$$

$$= \int_0^{\|u\|_\infty} \frac{D_H(t) + D_I(t)}{c_n m(t)^{1-2/n}} \, \mathrm{d}t.$$

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Proof: integration by parts. Equimeasurability gives $I(\omega_n) = T(\Omega)$, hence

$$G(0) = 0, \qquad G'(\omega_n) = \frac{1}{2n}, \qquad G(\omega_n) = T(\Omega) + \frac{\omega_n}{2(n+2)}.$$

Since u° is bounded, $sG'(s) \to 0$ as $s \downarrow 0$. Therefore

$$\begin{aligned}\int_0^{\omega_n} sG''(s)\,\mathrm{d}s &= [sG'(s)]_0^{\omega_n} - \int_0^{\omega_n} G'(s)\mathrm{d}s \\ &= \frac{\omega_n}{2n} - T(\Omega) - \frac{\omega_n}{2(n+2)} \\ &= \underbrace{\frac{\omega_n}{n(n+2)}}_{T(B_1)} - T(\Omega) = \delta_{\mathrm{SV}}(\Omega). \qquad \square\end{aligned}$$

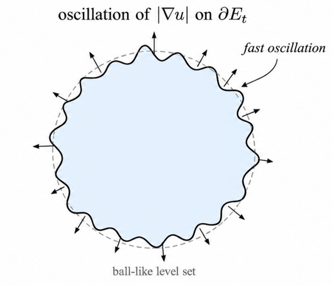
The torsional deficit separates two mechanisms

Theorem (Exact torsional deficit identity)

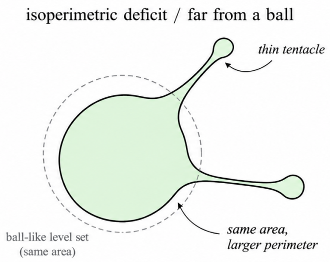
$$\delta_{\mathrm{SV}}(\Omega) = \int_0^{\|u\|_\infty} \frac{D_H(t) + D_I(t)}{c_n m(t)^{1-2/n}}\,\mathrm{d}t.$$

Since $D_H, D_I \geq 0$, the identity gives $\delta_{\mathrm{SV}}(\Omega) \geq 0$; this proves the Saint–Venant inequality.

D_H: variation of $|\nabla u|$ along $\partial^* E_t$



D_I: geometry of E_ρ via isoperimetric deficit



Differentiating the moving-ball distance

On $\partial^* E_\rho$, let ν_ρ be the outer unit normal and set

$$V_\rho \overset{\text{def.}}{=} \frac{w(\rho)}{|\nabla u|}, \qquad H_{z,\rho}(x) \overset{\text{def.}}{=} \frac{(x-z)\cdot \nu_\rho(x)}{\rho}, \qquad \overline{V}_\rho \overset{\text{def.}}{=} \frac{\mathrm{Per}(B_\rho)}{\mathrm{Per}(E_\rho)}.$$

V_ρ is the normal speed of the level set; $H_{z,\rho}$ is the normal speed of dilation about z.

For an optimal center z_ρ, for almost every ρ,

$$q'(\rho) \le \underbrace{\frac{n}{\rho} q(\rho)}_{I_1 \to \text{dilation}} + \underbrace{\int_{\partial^* E_\rho} |V_\rho - \overline{V}_\rho| \, \mathrm{d}\mathcal{H}^{n-1}}_{I_2 \to \text{speed oscillation}} + \underbrace{\int_{\partial^* E_\rho} |\overline{V}_\rho - H_{z_\rho,\rho}| \, \mathrm{d}\mathcal{H}^{n-1}}_{I_3 \to \text{geometric mismatch}}.$$

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- I_1 associated to $D_I(\rho)$ by quantitative isoperimetry
- I_2 associated to $D_H(\rho)$,
- I_3 associated to normal oscillation ... can it be controlled with D_I or D_H?

Strong isoperimetry controls the normal directions

Theorem (Fusco–Julin; Hensel–Laux — effective local form)

For $n \geq 2$ and $R_0 \geq 2$, there are computable constants $\varepsilon_{\mathrm{FJ}}, C_{\mathrm{FJ}} > 0$, depending on n, R_0, such that every set of finite perimeter $F \subset B_{R_0}$ with

$$|F| = \omega_n, \qquad \int_F x \,\mathrm{d}x = 0, \qquad \mathrm{Per}(F) - \mathrm{Per}(B_1) \leq \varepsilon_{\mathrm{FJ}},$$

satisfies

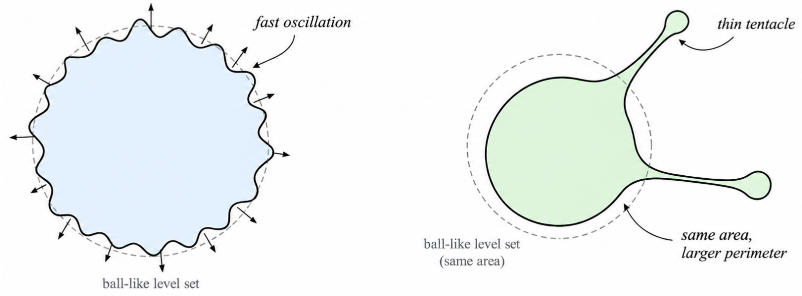
$$|F \,\Delta\, B_1|^2 + \int_{\partial^* F} \left| \nu_F - \frac{x}{|x|} \right|^2 \mathrm{d}\mathcal{H}^{n-1} \leq C_{\mathrm{FJ}} \big(\mathrm{Per}(F) - \mathrm{Per}(B_1) \big).$$

Hensel–Laux: a local calibration argument makes this effective.

Apply at the barycenter $\bar{z}_\rho$, then transfer to an optimal center z_ρ.[1]

[1] Fusco and Julin (2014), "A strong form of the quantitative isoperimetric inequality"; Hensel and Laux (2026), "Quantitative isoperimetry: a calibration argument".

Strong isoperimetry controls the normal directions



The strong normal estimate, paired with Cauchy–Schwarz for D_H, controls precisely the two variation terms in $q'(\rho)$:

$$\underbrace{\int_{\partial^* E_\rho} |V_\rho - \overline{V}_\rho| \,\mathrm{d}\mathcal{H}^{n-1}}_{\text{speed variation}} \le D_H(t(\rho))^{1/2}, \qquad \underbrace{\int_{\partial^* E_\rho} |\overline{V}_\rho - H_{z_\rho,\rho}| \,\mathrm{d}\mathcal{H}^{n-1}}_{\text{normal mismatch}} \le C D_I(t(\rho))^{1/2}.$$

The variations of the moving-ball distance are controlled by D_H and D_I.

The two defects control the growth of q

For $\rho \in [\rho_*, 1]$ with $D_I(t(\rho)) \le \eta_I$, we have

$$\int_{\partial^* E_\rho} |V_\rho - \overline{V}_\rho| \, \mathrm{d}\mathcal{H}^{n-1} \le D_H(t(\rho))^{1/2},$$

$$\int_{\partial^* E_\rho} |\overline{V}_\rho - H_{z_\rho,\rho}| \, \mathrm{d}\mathcal{H}^{n-1} \le C D_I(t(\rho))^{1/2}.$$

The differential estimate

$$q'(\rho) \le \frac{n}{\rho} q(\rho) + D_H(t(\rho))^{1/2} + C D_I(t(\rho))^{1/2}.$$

Also $q(\rho)^2 \le C(n) D_I(t(\rho))$ by quantitative isoperimetry.[1]

[1] Figalli, Maggi, and Pratelli (2010), "A mass transportation approach to quantitative isoperimetry".

Proof scheme

Saint–Venant in a bounded domain
$\Omega \subset B_R,\ |\Omega| = \omega_n$

Exact deficit identity

$$\delta_{\mathrm{SV}}(\Omega) = \int \frac{D_H(t) + D_I(t)}{c_n m^{1-2/n}(t)}\,\mathrm{d}t$$

Moving-ball estimates

$$q'(\rho) \le \frac{n}{\rho} q(\rho) + D_H^{1/2} + C D_I^{1/2}$$

Strong quantitative isoperimetry

Effective sharp Saint–Venant $\xrightarrow{\text{Kohler–Jobin}}$ **Effective sharp Faber–Krahn**

Proof in a bounded setting

The final argument now connects the end-point of the moving ball distance

$$q(1) = \omega_n \mathcal{A}(\Omega),$$

and the explicit Saint–Venant identity

$$\delta_{\mathrm{SV}}(\Omega) = \int_0^{\|u\|_\infty} \frac{D_H(t) + D_I(t)}{c_n m(t)^{1-2/n}}\,\mathrm{d}t = \int_0^1 \frac{D_H(t(\rho)) + D_I(t(\rho))}{n\omega_n \rho^{n-2}} w(t(\rho))\,\mathrm{d}\rho$$

connected via the upper bounds on q' and give the key H^1-type control on $q'_+ \overset{\text{def.}}{=} \max\{q', 0\}$:

The averaged estimate

$$\int_{\rho_*}^1 \left(q(\rho)^2 + (q'_+(\rho))^2\right)\,\mathrm{d}\rho \le C(n,R)\,\delta_{\mathrm{SV}}(\Omega). \qquad (q'_+ \in H^1)$$

A one-line endpoint argument finishes the proof

For every $s \in [\rho_*, 1]$, the fundamental theorem of calculus gives

$$q(1) = q(s) + \int_s^1 q'(\rho)\,\mathrm{d}\rho \le q(s) + \int_s^1 q'_+(\rho)\,\mathrm{d}\rho.$$

Average in s and apply Cauchy–Schwarz:

$$\begin{aligned} q(1) &\le \frac{1}{1-\rho_*}\int_{\rho_*}^1 q(s)\,\mathrm{d}s + \int_{\rho_*}^1 q'_+(\rho)\,\mathrm{d}\rho \\ &\le C\left(\|q\|_{L^2(\rho_*,1)} + \|q'_+\|_{L^2(\rho_*,1)}\right) \overset{(q'_+ \in H^1)}{\le} C(n,R)\sqrt{\delta_{\mathrm{SV}}(\Omega)}. \end{aligned}$$

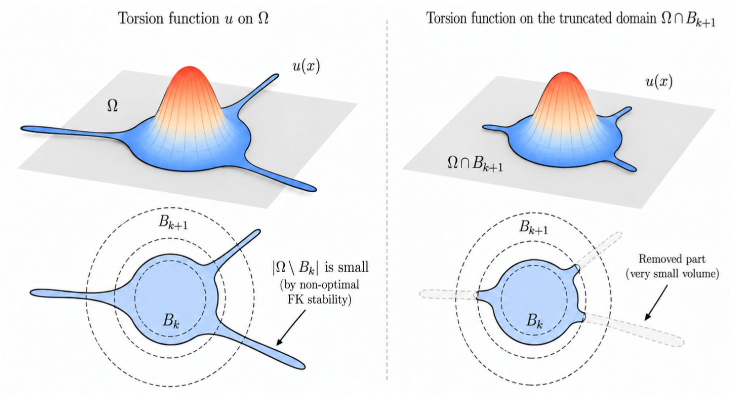
Since $q(1) = \omega_n \mathcal{A}(\Omega)$, we conclude

Sharp Saint–Venant stability in a bounded domain

$$\mathcal{A}(\Omega)^2 \le C(n,R)\,\delta_{\mathrm{SV}}(\Omega), \qquad \Omega \subset B_R.$$

Truncation removes the boundedness assumption

Brasco–De Philippis–Velichkov truncation:[1] cut where u is small, then renormalize the volume.



Why truncation is cheap

De Giorgi tail estimate

$$\|u\|_{L^\infty(\Omega \setminus B_{k+1})} \leq C(n)|\Omega \setminus B_k|^{1/n}.$$

A small geometric tail makes u uniformly small where we cut.

[1] Brasco, De Philippis, and Velichkov (2015), "Faber–Krahn inequalities in sharp quantitative form", Lemma 5.3.

Truncation removes the boundedness assumption

Brasco–De Philippis–Velichkov truncation:[1] cut where u is small, then renormalize the volume.

Since $\mathcal{A}(\Omega) \le 2$, we may assume $\delta_{\mathrm{SV}}(\Omega) \le \delta_*(n)$ after increasing the final constant.

Cubic Saint–Venant

$$\mathcal{A}(\Omega)^3 \le C_3(n)\delta_{\mathrm{SV}}(\Omega)$$

→

De Giorgi estimate

$$\|u\|_{L^\infty(\Omega\setminus B_{k+1})} \le C(n)|\Omega \setminus B_k|^{1/n} \le C(n)\delta_*(n)^{1/(3n)}$$

→

Choose $k = k(n)$

$$U \overset{\text{def.}}{=} \Omega \cap B_{k+1}, \quad \widetilde{\Omega} \overset{\text{def.}}{=} (|U|/\omega_n)^{-1/n} U, \quad \widetilde{\Omega} \subset B_{d(n)}$$

$$\delta_{\mathrm{SV}}(\widetilde{\Omega}) \approx \delta_{\mathrm{SV}}(\Omega), \qquad \mathcal{A}(\widetilde{\Omega}) \approx \mathcal{A}(\Omega)$$

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Theorem (Guerra–M.–Ramos)

For every $n \ge 2$ and every normalized open Ω, with computable dimensional constants,

$$\mathcal{A}(\Omega)^2 \le C_{\mathrm{SV}}(n)\delta_{\mathrm{SV}}(\Omega) \quad \overset{\text{Kohler–Jobin}}{\Longrightarrow} \quad \mathcal{A}(\Omega)^2 \le C_{\mathrm{FK}}(n)\delta_{\mathrm{FK}}(\Omega).$$

[1] Brasco, De Philippis, and Velichkov (2015), "Faber–Krahn inequalities in sharp quantitative form", Lemma 5.3.

Merci !