Phase-field approximation for 1-dimensional shape optimization problems

Joao Miguel Machado

PHASE-FIELD APPROXIMATION FOR 1-DIMENSIONAL SHAPE OPTIMIZATION PROBLEMS

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Abstract. In this paper we propose an unified framework for the phase field approximation of 1-dimensional shape optimization problems with connectedness constraints in any dimension. In particular, we focus on the average distance minimizers problem and the Wasserstein-$\mathscr{H}^1$ problem recently introduced in [15]. The scheme relies on the p-Ambrosio-Tortorelli energy and the diffuse connectedness functional proposed in [21] that penalizes how disconnected the level sets of phase fields are. We argue that choosing $p > d$, not only the optimal profiles coming from the Ambrosio Tortorelli term present sharper transitions, but it also allows us to control the level sets of phase fields, enabling the analysis of the connectedness functional. This leads to general $\Gamma - \liminf$ and $\limsup$ inequalities that are easily adaptable to prove Γ-convergence results for the average distance and Wasserstein-$\mathscr{H}^1$ problems.

Contents

1. Introduction

In these work we propose a unified phase field approximation for the Wasserstein-$\mathscr{H}^1$ problem, recently introduced in [15], and the average distance minimizers problem, see [30] for a review, in any dimension.

Henceforth, we let Ω be a compact and connected subset of $\mathbb{R}^d$ with nonempty interior and Lipschitz boundary, let $\mathscr{P}(\Omega)$ denote the set of probability measures over it. Given $\rho_0 \in \mathscr{P}(\Omega)$, the Wasserstein-$\mathscr{H}^1$ problem consists in finding the best approximation of ρ_0 among measures that are uniformly distributed over a 1-dimensional set Σ. It is given by

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the following variational problem

$$(W\mathscr{H}^1) \qquad \inf_{\Sigma \text{ connected}} W_q^q\left(\rho_0, \frac{1}{\mathscr{H}^1(\Sigma)}\mathscr{H}^1 \llcorner \Sigma\right) + \Lambda \mathscr{H}^1(\Sigma).$$

Here W_q denotes the q-Wasserstein distance between two probability measures, defined via the optimal transportation problem and which is known to metrize the weak convergence [35, 38], and $\mathscr{H}^1$ denotes the 1-Hausdorff measure [31].

Without the regularization term, this problem would be trivial as one could find a space-filling curve that makes the Wasserstein term converge to zero. On the other hand, the problem would also be trivial without the connectedness constraints, as one could approximate the measure ρ_0 with an empirical measure, paying nothing for the length term and making the Wasserstein distance arbitrarily small. It is proven in [15] that $(W\mathscr{H}^1)$ has a solution once Λ is sufficiently small and ρ_0 is smooth enough (does not give mass to 1D sets).

On the other hand, the average distance minimizers problem, first introduced in [14], is defined as

$$\text{(ADM)} \qquad \inf_{\Sigma \text{ connected}} \int_\Omega \operatorname{dist}(x, \Sigma)^q \mathrm{d}\rho_0(x) + \Lambda \mathscr{H}^1(\Sigma).$$

Now the measure ρ_0 represents a distribution of population over the domain Ω and the integral term models the average distance of this population to a transportation network Σ. The minimizers of (ADM) can then be interpreted as the best possible transportation network, in the sense that the average individual, distributed with law ρ_0, is closest to the transportation network.

As was pointed out in [10], the average distance functional can be expressed as

$$\int_\Omega \operatorname{dist}(x, \Sigma)^q \mathrm{d}\rho_0(x) = \inf_{\operatorname{supp}\nu \subset \Sigma} W_q^q(\rho_0, \nu),$$

so the average distance minimizers problem can be rewritten as

$$\text{(ADM)} \equiv \inf_{\Sigma \text{ connected}} \inf_{\operatorname{supp}\nu \subset \Sigma} W_q^q(\rho_0, \nu) + \Lambda \mathscr{H}^1(\Sigma).$$

This is the key observation that will allow us to propose a unified approach to study both problems.

In [15], a relaxation of $(W\mathscr{H}^1)$ was proposed as a minimization in the space of probability measures in the form of

$$(\overline{W\mathscr{H}^1}) \qquad \inf_{\nu \in \mathscr{P}(\Omega)} W_q^q(\rho_0, \nu) + \Lambda \mathcal{L}(\nu),$$

where $\nu \mapsto \mathcal{L}(\nu)$ is the *length functional* defined as

$$(1) \qquad \mathcal{L}(\nu) \overset{\text{def.}}{=} \inf\left\{\alpha \geq 0 : \alpha\nu \geq \mathscr{H}^1 \llcorner \operatorname{supp}\nu\right\},$$

the authors then show that it is the l.s.c. relaxation of the functional $\nu \mapsto \mathscr{H}^1(\Sigma)$ if ν is the probability measure uniformly distributed over a connected set Σ, *i.e.* $\nu = \dfrac{1}{\mathscr{H}^1(\Sigma)}\mathscr{H}^1 \llcorner \Sigma$, and $+\infty$ otherwise. As a consequence, one can rewrite the relaxation as a minimization in three variables: the measure ν, its support Σ and a new scalar variable α that measures the saturation of the density constraints $\alpha\nu \geq \mathscr{H}^1 \llcorner \Sigma$:

$$(2) \qquad (\overline{W\mathscr{H}^1}) \equiv \inf_{\Sigma \text{ connected}} \inf_{\substack{\alpha\nu \geq \mathscr{H}^1 \llcorner \Sigma \\ \operatorname{supp}\nu = \Sigma}} W_q^q(\rho_0, \nu) + \Lambda\alpha.$$

Under the assumptions for existence to $(W\mathscr{H}^1)$, the optimal α is given by $\mathscr{H}^1(\Sigma)$ and problem $(\overline{W\mathscr{H}^1})$ can formally be seen as (ADM) with additional density constraints.

Problems $(W\mathscr{H}^1)$ and (ADM) fall in the category of 1D shape optimization and are notoriously difficult to solve numerically in general. Perhaps the most famous of them is the Steiner tree problem [13]. It has many modern reformulations [33], one of which can be stated in the language of geometric measure theory as follows: given some Borel set K, we let $\mathscr{H}^1_S(K)$ denote the length of the minimal Steiner tree that connects K, therefore being defined as

$$\mathscr{H}^1_S(K) \stackrel{\text{def.}}{=} \inf \left\{ \mathscr{H}^1(\Sigma) : \quad K \subset \Sigma \text{ and } \Sigma \text{ is connected} \right\}, \tag{3}$$

and we let $\mathcal{S}(K)$ denote some tree that attains this value. From [33], the above minimization has a solution with possibly infinite length. In its original formulation, where K is a discrete set of points in $\mathbb{R}^2$, it can be proven that any optimal network is made of finitely many segments connected by triple junctions forming 120 degrees. It can therefore be seen as a combinatorial problem and is one of Karp's original NP-hard problems [28]. In the computer science and combinatorial optimization communities the natural approach to solving this type of problems is to resort to heuristic methods [39], and even in the calculations of variations this approach has been exploited in [1].

Another popular approach, which is variational in nature and indeed the one we shall adopt, is to resort to *phase field approximations*, that is a family of Sobolev functions whose level sets are good approximations of the set of small dimension we wish to approximate. The idea, originally from Modica and Mortola in [32] to study the Cahn-Hillard equations, and later of Ambrosio and Tortorelli in [2, 4] for the minimization of the Mumford-Shah problem [27, 29], is to find a family of elliptic functionals converging in the sense of Γ convergence ([11]) to the functional one wishes to minimize.

The Ambrosio-Tortorelli functional, see for instance the monograph [12],

$$\mathcal{AT}_p(\varphi_\varepsilon) = \frac{1}{\Lambda_{p,d}} \int_\Omega \left(\frac{\varepsilon^{p-d+1}}{p} |\nabla \varphi_\varepsilon|^p + \frac{\varepsilon^{-d+1}}{p'} (1-\varphi_\varepsilon)^2 \right) \mathrm{d}x$$

approximates formally the quantity $\mathscr{H}^1(\{\varphi = 0\})$, where $\Lambda_{p,d}$ is a renormalization constant, which will be obtained from an auxiliary variational problem see (16) later on, and p' is the conjugate exponent of p *i.e.* $\frac{1}{p} + \frac{1}{p'} = 1$. However, it does not penalize the connectedness of the set $\{\varphi = 0\}$. For more information on this functional see the discussion at the end of this Section and subsection 3.1.

In the original works from Ambrosio and Tortorelli about the Mumford-Shah functional, see [4, Chap. 6], the elliptic integrand was of the form

$$\varepsilon |\nabla \varphi_\varepsilon|^2 + \frac{(1-\varphi_\varepsilon)^2}{\varepsilon},$$

which coincides with ours if $p = d = 2$. We must emphasize that in the d-dimensional case, the Ambrosio-Tortorelli functional actually is meant to approximate $\mathcal{H}^{d-1}(J_u)$, where J_u is the jump set of an SBV function u. This way, the exponent $-(d-1)$ is meant to be the codimension of the type of structures we wish to approximate. This idea was proposed and further exploited for more general codimensions in [16].

Actually, the connectedness constraint in $(W\mathscr{H}^1)$(ADM) is the hardest to deal with phase field approximations, having only recently being treated in [10, 17, 37], where the authors' strategy to impose connectedness was to explore properties of the solutions, for instance a priori knowledge that certain points belong to the optimal networks. As we don't have such a priori knowledge for $(W\mathscr{H}^1)$ and (ADM), our strategy to impose such constraints will be to employ the connectedness functional proposed in [21] and later used to study several problems. For instance in [22] it was employed to minimize a variation of the Willmore energy with connectedness constraints, in [19] it is used to propose a phase-field approximation of the connected perimeter and in [20] it is used in the study of a liquid drop model with Coulomb type interaction. Their approach was to define the so-called *diffuse connectedness functional* as

$$\mathcal{C}_\varepsilon(\varphi_\varepsilon) \overset{\text{def.}}{=} \int_{\Omega\times\Omega} \beta_\varepsilon(\varphi_\varepsilon(x))\beta_\varepsilon(\varphi_\varepsilon(y)) d^{F_\varepsilon\circ\varphi_\varepsilon}(x,y)\mathrm{d}x\mathrm{d}y, \tag{4}$$

where $d^{F_\varepsilon\circ\varphi_\varepsilon}$ is a geodesic distance that penalizes the part of the path between its endpoints outside the level set $\{\varphi_\varepsilon \leq \varepsilon^s\}$, where s is a parameter that allows us to control the thickness of the transition regions from 0 to 1 of the optimal phase fields. The function β_ε is then designed to select only this level set on the integration over $\Omega\times\Omega$, for further details see Section 3.

The diffuse approximation results, in the sense of Γ-convergence, that we prove in this work are formulated with respect to the following functionals:

$$\mathcal{AD}_\varepsilon(\nu_\varepsilon,\varphi_\varepsilon) \overset{\text{def.}}{=} \begin{cases} W_q^q(\rho_0,\nu_\varepsilon) + \Lambda\mathcal{AT}_p(\varphi_\varepsilon) + \dfrac{1}{\varepsilon^\kappa}\mathcal{C}_\varepsilon(\varphi_\varepsilon) & \nu_\varepsilon \in \mathscr{P}(\Omega), \\ \qquad + \dfrac{1}{\varepsilon^\ell}\displaystyle\int_\Omega \varphi_\varepsilon \mathrm{d}\nu_\varepsilon, & \varphi_\varepsilon \in 1 + W_0^{1,p}(\Omega) \\ +\infty, & \text{otherwise,} \end{cases} \tag{5}$$

the *diffuse average distance* functional, and

$$\mathcal{WH}^1_\varepsilon(\alpha_\varepsilon,\nu_\varepsilon,\varphi_\varepsilon) \overset{\text{def.}}{=} \begin{cases} W_q^q(\rho_0,\nu_\varepsilon) + \Lambda\alpha_\varepsilon + \dfrac{1}{\varepsilon}\left\|\alpha_\varepsilon\nu_\varepsilon - \mu_\varepsilon\right\|^2_{L^2(\Omega)} & \alpha_\varepsilon \geq 0, \\ & \nu_\varepsilon \in \mathscr{P}(\Omega) \\ \qquad + \dfrac{1}{\varepsilon^\kappa}\mathcal{C}_\varepsilon(\varphi_\varepsilon) + \dfrac{1}{\varepsilon^\ell}\displaystyle\int_\Omega \varphi_\varepsilon \mathrm{d}\nu_\varepsilon, & \varphi_\varepsilon \in 1 + W_0^{1,p}(\Omega), \\ +\infty, & \text{otherwise.} \end{cases} \tag{6}$$

where the measure $\mu_\varepsilon = \mu_\varepsilon(\varphi_\varepsilon)$ is the diffuse transition measure and is defined as

$$\mu_\varepsilon \overset{\text{def.}}{=} \frac{1}{\Lambda_{p,d}}\left(\frac{\varepsilon^{p-d+1}}{p}|\nabla\varphi_\varepsilon|^p + \frac{\varepsilon^{-d+1}}{p'}(1-\varphi_\varepsilon)^2\right)\mathcal{L}^d \llcorner \Omega, \tag{7}$$

where $\Lambda_{p,d}$ depends only on p and d, see Section 3 for further details and properties about these measures and, in particular the variational interpretation of this constant.

Before stating our results, we make the following hypothesis that will be assumed without statement throughout this work.

(H1) $\Omega \subset \mathbb{R}^d$ is a compact, connect set with Lipschitz boundary and such that $\overline{\mathrm{int}\Omega} = \Omega$, and it is *star-shaped* that is there exists $x_\star \in \operatorname{int}\Omega$ such that $\lambda x_\star + (1-\lambda)x \in \operatorname{int}\Omega$ for any $x \in \Omega$ and $\lambda \in (0,1)$.

Hypothesis (H1) is to prevent the loss of mass, due to concentration on the boundary, while passing to the limit in the weak-$\star$ topology of probability measures.

The first result concerns the approximation for the average distance minimizers problem, in the spirit of the results found in [10] for instance. The difference of our approach is that, due to the diffuse connectedness functional, we do not need the a priori knowledge that the optimal network contains any specific point.

Theorem 1.1. *Assume that $\ell > s > 1$, $\kappa > (2d+1)\frac{(s+1)p-d+1}{p-d}$ and that $p > d \geq 2$. Then it holds that*

$$\mathcal{AD}_\varepsilon \xrightarrow[\varepsilon\to 0]{\Gamma} \mathcal{AD}(\nu,\varphi) \overset{\text{def.}}{=} \begin{cases} W_q^q(\rho_0,\nu) + \Lambda\mathscr{H}^1_S(\operatorname{supp}\nu), & \nu \in \mathscr{P}(\Omega),\ \varphi \equiv 1, \\ +\infty, & \textit{otherwise,} \end{cases}$$

where $\mathscr{H}^1_S(\operatorname{supp}\nu)$ is the length of the minimal Steiner tree connecting $\operatorname{supp}\nu$*, defined in* (3)*. The Γ-convergence holds in the strong topology of L^2 and weak-$\star$ topology of $\mathscr{P}(\Omega)$.*

In addition, let $(\nu_\varepsilon, \varphi_\varepsilon)_{\varepsilon>0}$ *be a family of minimizers of* $\mathcal{AD}_\varepsilon$*, it admits a cluster point* $(\nu, \varphi \equiv 1)$*, which then achieves the infimum and*

$$\min_{\Sigma} \text{(ADM)} = \min_{(\nu,\varphi)} \mathcal{AD}(\nu, \varphi),$$

and it holds that

- Σ *is a minimizer of* (ADM) *if, and only if, it is a minimal Steiner tree of* $\operatorname{supp}\nu$*, for some* ν *minimizer of* $\mathcal{AD}$*;*
- ν *is a minimizer of* $\mathcal{AD}$ *if, and only if, it can be written as* $\nu = (\pi_\Sigma)_\sharp \rho_0$*, where* π_Σ *is a measurable selection of the projection operator onto some* Σ *minimizer of* (ADM)*.*

It is important to point out that, given an optimal network Σ for (ADM), the measure $\nu = (\pi_\Sigma)_\sharp \rho_0$, that is the minimizers of $\mathcal{AD}$, carries important information about the topology of Σ. Indeed, it was shown in [14], see also [36], that points $y \in \Sigma$ such that $\nu(\{y\}) > 0$ are either end-points or corner points of Σ, see also the survey [30]. Therefore, the approximation we propose carries the information of the optimal network though the level sets of phase fields, and of the expected topology, though the approximations of the measure ν.

Our second Γ convergence result concerns the relaxed problem ($\overline{W\mathscr{H}^1}$). Since the energy in ($W\mathscr{H}^1$) is not l.s.c., as seen as a functional in $\mathscr{P}(\Omega)$, we cannot hope to prove a Γ-convergence result for it, since Γ limits always are l.s.c. in the topology inducing the Γ convergence. What we strive instead, is to approximate the relaxed problem, so that under the assumptions on ρ_0 that guaranties existence for ($W\mathscr{H}^1$), see [15], any cluster points of minimizers of $\mathcal{WH}^1_\varepsilon$ will also minimize the original problem ($W\mathscr{H}^1$).

Theorem 1.2. *Assume that* $\ell > s > 1$*,* $\kappa > (2d+1)\frac{(s+1)p-d+1}{p-d}$ *and* $p > d \geq 2$*. Then it holds that*

$$\mathcal{WH}^1_\varepsilon \xrightarrow[\varepsilon\to 0]{\Gamma} \overline{\mathcal{WH}^1}(\alpha, \nu, \varphi) \overset{\text{def.}}{=} \begin{cases} W_q^q(\rho_0, \nu) + \Lambda\mathcal{L}(\nu), & \nu \in \mathscr{P}(\Omega),\ \alpha \geq \mathcal{L}(\nu), \\ +\infty, & \textit{otherwise,} \end{cases}$$

the Γ*-convergence being held in* $\mathbb{R}$*, the strong topology of* L^2 *and weak-*$\star$ *topology of* $\mathscr{P}(\Omega)$*.*

In addition, whenever ρ_0 *does not charge countably* $\mathscr{H}^1$*-rectifiable sets, if* $(\alpha_\varepsilon, \nu_\varepsilon, \varphi_\varepsilon)_{\varepsilon>0}$ *is a sequence of minimizers of* $\mathcal{WH}^1_\varepsilon$*, then it has a cluster point* $(\alpha, \nu, \varphi \equiv 1)$ *of the form*

$$\alpha = \mathscr{H}^1(\Sigma),\ \nu = \frac{1}{\mathscr{H}^1(\Sigma)}\mathscr{H}^1 \llcorner \Sigma,\ \textit{where } \Sigma \textit{ is connected } \mathscr{H}^1\textit{-rectifiable},$$

so that Σ *minimizes* ($W\mathscr{H}^1$)*.*

A few remarks about Theorems 1.1 and 1.2 and their proofs are in order. First of all, the formal relation between ($\overline{W\mathscr{H}^1}$) and (ADM) becomes more evident from the proposed phase field approximations. Indeed, the proofs of our Γ-convergence approximations only differ on how we deal with the support of the measures ν. In the average distance minimizers problem we do not need to control it as much as in the Wasserstein-$\mathscr{H}^1$ problem, since in the former we only need the support to be contained in a 1-dimensional set (the support doesn't even need to be rectifiable), in the latter we must distribute a minimal amount of mass everywhere.

The lower bound for κ is not very encouraging for numerics since the quantity $\varepsilon^{-\kappa}$ can very quickly exceed machine precision with not so small values for ε. In other models, for instance the connected Willmore energy, Γ convergence results have been achieved with $\kappa = 2$, see [22]. Since in our problems the phase fields approximate 1-dimensional structures, instead of sets of finite perimeter as in [19, 22], it is expected the value for κ in our problems to be larger. That said, the argument in Theorem 3.8 is probably not optimal and there might be another argument that gives a smaller bound for κ. However, in practice numerical experiments have shown to work with different constants penalizing $\mathcal{C}_\varepsilon$, [19].

Finally, from a numerical point of view, the case $p = 2$ in the Ambrosio-Tortorelli term is much more convenient. However, in this work we need to assume

$$p > d \text{ in the functional } \mathcal{AT}_p. \tag{8}$$

This will imply that phase-fields with finite energy belong in the space $W^{1,p}(\Omega)$, and from standard Sobolev injections they must be Hölder continuous. Not only the enhanced regularity is of paramount importance in controlling the small level sets of the phase fields, synergizing well with the connectedness functional in the general $\Gamma - \liminf$ inequality, see Thm. 3.8, but it also helps in the matter of existence of solutions for the sequence of approximated problems.

In [10], the matter of existence was solved by adding a penalization of $\|\nabla\varphi_\varepsilon\|_{L^p(\Omega)}$, having the same effect, but possibly affecting the numerics, as it was only a parasite term in the minimization and not contributing to the approximation of the length. In [8,9], to approximate the Steiner tree problem, the question of existence was dealt with by means of a regularization with a term reminiscent of the Willmore energy [40,41], forcing phase fields to be in $W^{2,2}(\Omega)$. A disadvantage of our approach is that the first variation of our energy will have a p-Laplacian term. On the other hand, computing variations for the Willmore energy require the solution of a fourth order PDE. It is not clear then which approach would be more computationally demanding and extensive numerical experiments and testing are in order.

In addition to the aforementioned reasons, for the case $p > d = 2$ case, we shall also see in Proposition 3.1 and Theorem 3.4 that it provides a better optimal profile, in the sense that its transition width is of the order $\frac{p}{p-2}\varepsilon$, as opposed to $\varepsilon \log \varepsilon$ in the case $p = 2$, see [10]. As a result, one can expect sharper transitions with $p > 2$, and therefore, better qualitative results. This intuition is corroborated in Section 3.1, see in particular Figures 3.1 and 3.1. We also characterize the optimal profiles for $d \geq 3$, but in this case its computation is no longer explicit.

This article is organized as follows: in Section 2 we review the basic terminology and known mathematical results we shall employ, namely optimal transportation and geometric measure theory. A special care is dedicated to the properties of rectifiable sets and their blow-ups. In Section 3 we start by proving general results about the Ambrosio-Tortorelli and the connectedness functional to then discuss the central results in this paper, Theorems 3.8 and 3.9. In these results we give the major arguments that are then adapted in Section 4 to the Γ-convergence for both the average distance minimizers and the $W - \mathscr{H}^1$ problems. In Section 5 we give our concluding remarks.

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2. Preliminaries

In this section we introduce the mathematical tools to be used throughout the paper. We start by defining the optimal transportation problem used to define the Wasserstein distance. In the sequel, we discuss the class of connected sets with finite $\mathscr{H}^1$ measure and argue that these sets are called *rectifiable*, they are almost the countable union of Lipschitz graphs. Finally, we introduce the concept of Γ convergence.

2.1. **Spaces of probability measures and Wasserstein distances.** Given $\Omega \subset \mathbb{R}^d$ compact, the space of probability measures over Ω, $\mathscr{P}(\Omega)$, is defined as the subspace of $\mathcal{M}_+(\Omega)$, the positive and bounded Radon measures, that have unitary total mass, [4]. From the Riesz representation theorem, the space of bounded Radon measures $\mathcal{M}_b(\Omega)$ is the topological dual of the space of continuous functions vanishing at the boundary $C_0(\Omega)$, and as such, a sequence of measures $\mu_n \xrightarrow[n\to\infty]{\star} \mu$ if and only if

$$\int_\Omega \phi \mathrm{d}\mu_n \xrightarrow[n\to\infty]{} \int_\Omega \phi \mathrm{d}\mu, \text{ for all } \phi \in C_0(\Omega).$$

This weak-$\star$ convergence can be metrized by the so called Wasserstein distances, see [35, Thm. 5.9], which are defined through the value of the optimal transportation problem as

$$W_q^q(\mu,\nu) \stackrel{\text{def.}}{=} \inf_{T_\sharp\mu=\nu} \int_\Omega |x - T(x)|^q \mathrm{d}\mu = \min_{\gamma\in\Pi(\mu,\nu)} \int_{\Omega\times\Omega} |x-y|^q \mathrm{d}\gamma. \tag{9}$$

In (9), the infimum is known as the Monge's formulation and is taken over all the Borel maps T such that the pushforward measure, defined as $T_\sharp\mu(A) \stackrel{\text{def.}}{=} \mu(T^{-1}(A))$ for any Borel set A, coincides with the measure ν. The minimum in (9), called the Kantorovitch formulation, is taken over all probability measures over the product space whose marginal are μ and ν *i.e.*

$$\Pi(\mu,\nu) \stackrel{\text{def.}}{=} \left\{\gamma \in \mathscr{P}(\Omega\times\Omega) : (\pi_0)_\sharp\gamma = \mu,\ (\pi_1)_\sharp\gamma = \nu\right\}.$$

The Kantorovitch formulation is a relaxation of the first one and its value coincides with Monge's whenever μ does not have atoms [34]. It also always admits minimizers, which is not true in general for the formulation with transportation maps, but whenever $\mu \ll \mathcal{L}^d$, μ is absolutely continuous w.r.t. the Lebesgue measure and we write $\mu \in \mathscr{P}_{ac}(\Omega)$, any optimal plan is of the form $\gamma = (\mathrm{id}, T)_\sharp\mu$ and hence Monge's formulation has a solution. For further details on optimal transport and its applications, the reader is referred to [3,35,38].

2.2. **Rectifiable sets.** We recall that $\mathscr{H}^1$ denotes the 1 dimensional Hausdorff measure over $\mathbb{R}^d$, see [4, Chap. 2]. A Borel set Σ is said to be $\mathscr{H}^1$-countably rectifiable, or simply 1-rectifiable, if there exists countably many Lipschitz functions $f_n : [0,1] \to \mathbb{R}^d$ such that

$$\mathscr{H}^1\left(\Sigma \setminus \bigcup_{n\in\mathbb{N}} f_n([0,1])\right) = 0. \tag{10}$$

It was proven in the pioneering works of Besicovitch [7, Chap. 8] that connected sets with finite $\mathscr{H}^1$-measure are countably $\mathscr{H}^1$-rectifiable. His original proof consists in a clever greedy algorithm that is able to cover $\mathscr{H}^1$-a.a. of such a set with countably many iterations, see also [5, Thm. 4.4.8] for a more modern flavor.

We can also define a metric in the class of closed subsets of Ω. For A, B closed sets, we define the Hausdorff distance as

$$d_H(A,B) := \max\left\{\sup_{a\in A} \mathrm{dist}(a,B), \sup_{b\in B} \mathrm{dist}(b,A)\right\}. \tag{11}$$

This distance has the useful property of preserving connectedness: if a sequence of connected sets $(A_n)_{n\in\mathbb{N}}$ converges in the Hausdorff sense to A, then A is also connected. It is also compact, Blaschke's Theorem [4, Theorem 6.1], states that whenever Ω is compact any sequence of sets has a subsequence, convergent in the Hausdorff distance.

The final ingredient that we will need in our analysis is the notion of *approximate tangent space*. It turns out that rectifiable sets are very similar objects to smooth manifolds in the sense that they admit a weak measure-theoretic notion of tangent space defined with blow-ups.

Given a 1-rectifiable set Σ, set $\mu = \mathscr{H}^1 \llcorner \Sigma$ and define the family of blow-up measures

$$\mu_r := r^{-1}\Phi^{x,r}_\sharp \mu = \mathscr{H}^1 \llcorner \left(\frac{\Sigma - x}{r}\right), \text{ for } \Phi^{x,r} := \frac{\mathrm{id} - x}{r}. \tag{12}$$

The blow-up Theorem, see [31, Theorem 10.2], states that for $\mathscr{H}^1$-a.e. $x \in \Sigma$, there exists an unique 1-plane, which we call $T_x\Sigma$, such that $\mu_r \xrightarrow[r\to 0]{\star} \mathscr{H}^1 \llcorner T_x\Sigma$. In particular, it can be written as $T_x\Sigma = \mathbb{R}\tau$, for a unique vector $\tau \in \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$. Hence, for a.e. $x \in \Sigma$ we can define the approximate tangent space of Σ at x as the plane $T_x\Sigma$. For any point $x \in \Sigma$ such that the approximate tangent space $T_x\Sigma$ is well-defined, we shall also say that Σ is flat at x.

We summarize this discussion in the following Theorem, in which we also introduce a flatness result for points $x \in \Sigma$ whose blow-up is a straight line that is carried to sequences of sets which are converging to Σ in the Hausdorff topology.

Theorem 2.1. *Let $\Sigma \subset \mathbb{R}^d$ be closed and connected with $\mathscr{H}^1(\Sigma) < +\infty$, then the following hold.*

(1) For $\mathscr{H}^1$-a.e. $x \in \Sigma$ it holds that

$$\mathscr{H}^1 \llcorner \left(\frac{\Sigma - x}{r}\right) \xrightarrow[r\to 0]{\star} \mathscr{H}^1 \llcorner T_x\Sigma,$$

where $T_x\Sigma$ is the approximate tangent space of Σ at x, and it also holds that

$$\frac{d_H\left((\Sigma - x)\cap B_r, T_x\Sigma \cap B_r\right)}{r} = d_H\left(\frac{\Sigma - x}{r} \cap B_1, T_x\Sigma \cap B_1\right) \xrightarrow[r\to 0]{} 0,$$

where the balls whose center are omitted are centered at 0.

In the sequel we assume that Σ is flat at x_0 with approximate tangent space $T_{x_0}\Sigma = \mathbb{R}\tau$, for $\tau \in \mathbb{S}^{d-1}$ and let π_τ denote the projection onto it.

(2) For $1/2 < \delta < 1$, there is some r_0 such that

$$[-\delta r, \delta r]\tau \subset \pi_\tau(\Sigma \cap B_r(x_0)), \text{ for all } r < r_0.$$

That is, for any $t \in [-\delta r, \delta r]$ there is $x \in \Sigma \cap B_r(x_0)$ such that $\langle \tau, x\rangle = t$. In addition, x belongs to the connected component of $\Sigma \cap B_r(x_0)$ that contains x_0.

(3) Let Σ_r denote the connected component of $\Sigma \cap B_r(x_0)$ which contains x_0, then

$$\frac{\Sigma_r - x_0}{r} \xrightarrow[r\to 0]{d_H} [-\tau, \tau].$$

(4) Let $(\Sigma_\varepsilon)_{\varepsilon>0}$ be a family of connected sets such that $\Sigma_\varepsilon \xrightarrow[\varepsilon\to 0]{d_H} \Sigma$. Then for $1/2 < \delta < 1$, there are r_0 and ε_0 such that, if $r < r_0$ and $\varepsilon < \varepsilon_0$, for each $t \in (-\delta r, \delta r)$, there exists

$$x \in \Sigma_\varepsilon \cap B_r(x_0), \text{ such that } \pi_\tau(x) = x_0 + t\tau, \tag{13}$$

except in a set that is either a singleton, or a connected interval $(a_\varepsilon, b_\varepsilon)$ such that $b_\varepsilon - a_\varepsilon \leq 2d_H(\Sigma_\varepsilon, \Sigma)$.

Proof. Since Σ is connected and $\mathscr{H}^1(\Sigma) < \infty$, it is countably $\mathscr{H}^1$-rectifiable, so the convergence of the family of blow-up measures in item (1) is a classical result, see for instance [31, Chap. 10]. To check the Hausdorff convergence of the blow-up family of sets, notice that from homogeneity of the distance in $\mathbb{R}^d$ it holds that

$$\frac{d_H\left((\Sigma - x)\cap B_r, T_x\Sigma \cap B_r\right)}{r} = d_H\left(\frac{\Sigma - x}{r} \cap B_1, T_x\Sigma \cap B_1\right)$$

and the RHS converges to zero as $r \to 0$ from [15, Lemma 2.5].

Item (2) is proven in [10] in the case $d = 2$, for completeness we prove it here in $\mathbb{R}^d$. Using item (1), take r_0 small enough such that

$$\begin{gathered} d_H\left(\frac{\Sigma \cap B_r(x_0) - x_0}{r}, [-\tau, \tau]\right) \leq (1-\delta), \\ \Sigma \cap B_{\delta r}(x_0) \subset [-\delta r, \delta r]\tau + B_{(1-\delta)r}(x_0). \end{gathered} \tag{14}$$

Therefore, there must be points $z_+, z_- \in (\Sigma \cap B_r(x_0) - x_0)$ such that $|z_\pm - (\pm\tau)| \leq (1-\delta)r$ and paths $\gamma_\pm \subset [-\delta r, \delta r]\tau + B_{(1-\delta)r}(x_0)$ connecting x_0 and $z_\pm$. Therefore, we must have that $\pi_\tau(z_+) \geq \delta r$ and $\pi_\tau(z_-) \geq -\delta r$ so that $\pi_\tau(\gamma_+)$ (resp. $\pi_\tau(\gamma_-)$) must be a connected set containing $x_0 + [0, \delta r]$ (resp. $x_0 + [-\delta r, 0]$).

Using property (2) we prove item (3) as follows: suppose that $(r_n)_{n\in\mathbb{N}}$ is an infinitesimal sequence such that

$$L_n \overset{\text{def.}}{=} \frac{\Sigma_{r_n} - x_0}{r_n} \xrightarrow[n\to\infty]{d_H} L \subset [-\tau, \tau],$$

such subsequence exists from Blaschke's Theorem. Given $\delta > 0$, let $r_0 > 0$ be the radius obtained from item (2). Let ρ_k be such that for $r < \rho_k$

$$\frac{\Sigma_r - x_0}{r} \subset \frac{\Sigma \cap B_r(x_0) - x_0}{r} \subset B([-\tau, \tau], 1/k).$$

So we can construct a subsequence $(r_{n_k})_{k\in\mathbb{N}}$ such that $r_{n_k} < \rho_k$ and using item (2), assuming k sufficiently large, for any $t \in [-\delta, \delta]$ we can also find a point

$$x_{n_k} \in \frac{\Sigma_{r_{n_k}} - x_0}{r_{n_k}} \text{ s.t. } \langle x_{n_k}, \tau\rangle = t, \ \operatorname{dist}(x_{n_k}, [-\tau, \tau]) < 1/k.$$

As a consequence $x_{n_k} \xrightarrow[k\to\infty]{} t\tau$, which implies that

$$[-\delta, \delta]\tau \subset L \subset [-\tau, \tau],$$

for any $0 < \delta < 1$, so that as L is concluded, we conclude that $L = [-\tau, \tau]$. Since any cluster point of $r^{-1}(\Sigma_r - x_0)$ equals to L, then the entire family must converge to it.

Item (4) can be interpreted as a partial transfer of property (2) to any sequence of connected sets Σ_ε converging to Σ, up to a small set that can be quantified. From the Hausdorff convergence in item (1), we can choose r_0 such that for $r < r_0$ we have

$$\Sigma \cap B_{\delta r}(x_0) \subset x_0 + B_{\frac{r\delta'}{2}}([-\tau, \tau]), \text{ with } \delta' = \sqrt{1-\delta^2}.$$

In addition, the cylinder

$$C_{\delta,r}(x_0) \overset{\text{def.}}{=} \left\{ x : \begin{array}{c} |\pi_1(x - x_0)| < \delta r \\ |\pi_{\tau^\perp}(x - x_0)| < \delta' r \end{array} \right\} \text{ is contained in the ball } B_{\delta r}(x_0).$$

Suppose by contradiction that the set of points that do not satisfy (13) is disconnected and take tow points t and t' in two distinct connected components. From property (2), between these sections there is a point of Σ, *i.e.* there exists $y \in \Sigma$ inside the smaller cylinder $\pi_\tau^{-1}((t'r, tr)\tau) \cap C_{\delta,r}(x_0)$.

From the Hausdorff convergence of Σ_ε to Σ, for ε small enough, there exists $y_\varepsilon \in \Sigma_\varepsilon$ that can be made arbitrarily close to y taking ε small enough, so that

$$y_\varepsilon \in \pi_\tau^{-1}((t'r, tr)\tau) \cap C_{\delta,r}(x_0), \text{ and } \Sigma_\varepsilon \cap B_r(x_0) \subset x_0 + B_{\delta' r}([-\tau, \tau]).$$

But then as Σ_ε is connected, there is a path connecting y_ε to some point of Σ_ε outside $B_r(x_0)$. This path must then intersect $(\pi_\tau)^{-1}(\{t'\tau, t\tau\})$, which contradicts the fact that t and t' do not satisfy (13).

This proves that if the set of values $t \in (-\delta r, \delta r)$ not satisfying (13) is either a singleton or a connected interval. In the latter case, assume it to be given by $(a_\varepsilon, b_\varepsilon) \subset (-\delta r, \delta r)$; suppose that $b_\varepsilon - a_\varepsilon > 2d_H(\Sigma_\varepsilon, \Sigma)$. In this case take

$$y \in \Sigma \cap B_r(x_0), \text{ such that } \langle \tau, y_0\rangle = \frac{b_\varepsilon + a_\varepsilon}{2} > d_H(\Sigma_\varepsilon, \Sigma),$$

such point exists from item (2). This means that the closest point of Σ_ε to y is at distance bigger than $d_H(\Sigma_\varepsilon, \Sigma)$, which contradicts the definition of the Hausdorff distance between Σ_ε and Σ. $\square$

2.3. **Γ-convergence.** The notion of Γ-convergence was introduced by de Giorgi in order to have good properties concerning the limits of minimizers of variational problems, see for instance [18] for a compreensive monograph about this theory.

Definition 2.2. Let (X, d) be a complete metric space. We say that a sequence of functionals $F_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$ Γ-converges to F, and we write $F_\varepsilon \xrightarrow[\varepsilon\to 0]{\Gamma} F$ if the two following conditions hold

- $\underline{\Gamma - \liminf}$**:** for every family $x_\varepsilon \to x$ in X, it holds that
$$F(x) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon).$$
- $\underline{\Gamma - \limsup}$**:** for every $x \in X$, there is a sequence $x_\varepsilon \to x$ such that
$$\limsup_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \leq F(x).$$

The fundamental property that makes Γ-convergence an interesting notion for variational problems is the fact that cluster points of minimizers of a sequence of minimizers of F_ε are minimizers of F.

Lemma 2.3 ([6])**.** *Let F_ε be a family of functionals over a metric space (X, d), which Γ-converging to F. If $(x_\varepsilon)_{\varepsilon>0}$ is such that*
$$x_\varepsilon \in \operatorname*{argmin}_X F_\varepsilon, \ \textit{for all } \varepsilon > 0,$$
then if x is a cluster point of this family, then $x \in \operatorname*{argmin}_X F$.

This property motivates the approximation of 1-rectifiable sets Σ, that are hard to implement computationally, with the level sets of Sobolev functions.

3. The Γ-convergence: the general theory

In this section we write general results about the interplay between the Ambrosio-Tortorelli term and connectedness functional in Γ-convergence. We start by studying $\mathcal{AT}_p$ and $\mathcal{C}_\varepsilon$ separately in paragraphs 3.1 and 3.2, and finally we give flexible Theorems in Subsection 3.3 that give conditions under which these functionals behave well together, allowing to prove both Γ-convergence Theorems 1.1 and 1.2. Hopefully this modular presentation will be helpful in the analysis of phase-field approximations for other problems in the future.

3.1. **Properties of $\mathcal{AT}_p$.** In this section we discuss the individual properties of the Ambrosio-Tortorelli type functional defined as

$$\mathcal{AT}_p(\varphi_\varepsilon) \stackrel{\text{def.}}{=} \frac{1}{\Lambda_{p,d}} \int_\Omega \left(\frac{\varepsilon^{p-d+1}}{p} |\nabla \varphi_\varepsilon|^p + \frac{\varepsilon^{-d+1}}{p'} (1-\varphi_\varepsilon)^2 \right) \mathrm{d}x. \tag{15}$$

As we have chosen $p > d$, see the discussion surrounding this choice in the introduction, if $\mathcal{AT}_p(\varphi_\varepsilon) < \infty$ the family of phase fields φ_ε belongs to the Sobolev space $W^{1,p}(\Omega)$, and since we assume from the start that Ω has a Lipschitz boundary, from the classical Sobolev injection $W^{1,p}(\Omega) \hookrightarrow C^{0,\beta}(\overline{\Omega})$, φ_ε is β-Hölder continuous for $\beta = 1 - \frac{d}{p}$.

We recall the definition of the diffuse transition measure defined in (7) as
$$\mu_\varepsilon \stackrel{\text{def.}}{=} \frac{1}{\Lambda_{p,d}} \left(\frac{\varepsilon^{p-d+1}}{p} |\nabla \varphi_\varepsilon|^p + \frac{\varepsilon^{-d+1}}{p'} (1-\varphi_\varepsilon)^2 \right) \mathcal{L}^d \llcorner \Omega.$$

We shall see that not only $\mathcal{AT}_p(\varphi_\varepsilon)$ approximates the quantity $\mathscr{H}^1(\Sigma)$, but one can also find a good family of phase fields φ_ε such that $\mu_\varepsilon \xrightarrow[\varepsilon\to\infty]{\star} \mathscr{H}^1 \llcorner \Sigma$, whenever Σ is a connected set. With this goal, let us start with a simple example, of approximating a segment $L = [0, 1]e_d$ in $\mathbb{R}^d$. By symmetry, it is natural to expect that a radially symmetric profile around the d-axis would suffice. This motivates the following $(d-1)$-variational problem

$$\Lambda_{p,d} \overset{\text{def.}}{=} \min\left\{ \mathcal{C}_{p,d}(u) \overset{\text{def.}}{=} \int_{\mathbb{R}^{d-1}} \left(\frac{1}{p}|\nabla u|^p + \frac{1}{p'}(1-u)^2\right) \mathrm{d}x : \begin{array}{c} u(0)=0 \\ \nabla u \in L^p(\mathbb{R}^{d-1}) \\ 1-u \in L^2(\mathbb{R}^{d-1}) \end{array} \right\}, \tag{16}$$

which is inspired on the analysis of [16] and will be used in the proofs of the Γ-liminf and Γ-limsup. Clearly, functions u with finite energy above are equal to 1 at infinity. Moreover, we can relate solutions of (16) with the following problem in $\mathbb{R}$

$$\lambda_{p,d} \overset{\text{def.}}{=} \min\left\{ c_{p,d}(f) \overset{\text{def.}}{=} \int_0^{+\infty} t^{d-2}\left(\frac{1}{p}|f'|^p + \frac{1}{p'}(1-f)^2\right) \mathrm{d}t : \begin{array}{c} f(0)=0, \\ f \in AC^p(\mathbb{R}_+) \end{array} \right\}, \tag{17}$$

where $AC^p(\mathbb{R}_+)$ denotes the space of p-absolutely continuous curves.

It is harder to apply the Direct method to this second problem since the term t^{d-2} gets in the way of bounding the L^p-norm of the velocities, but we manage to derive existence and uniqueness for (17) from (16). In dimension $d=2$, it can be solved explicitly without resorting to the Euler-Lagrange equations.

Proposition 3.1. *For any $p \geq 2$, the variational problem* (16) *admits a unique minimizer, which is radially symmetric of the form $u(x) = f_p(|x|)$, where $f_p : \mathbb{R}_+ \to [0,1]$ is the unique non-decreasing Hölder continuous minimizer of* (17)*, and we have that*

$$\Lambda_{p,d} = \sigma_{d-2}\lambda_{p,d}, \ \text{ where } \sigma_{d-2} = \mathscr{H}^{d-2}(\mathbb{S}^{d-2})$$

is the area of the $d-2$ unit sphere in $\mathbb{R}^{d-1}$.

In addition, for the case $p > d = 2$ the optimal profile f_p is given by

$$f_p(t) \overset{\text{def.}}{=} \begin{cases} 1 - \left(1 - \frac{p-2}{p}t\right)^{\frac{p}{p-2}}, & 0 \leq t \leq \frac{p}{p-2}, \\ 1, & t \geq \frac{p}{p-2}, \end{cases}$$

so that the value can be computed explicitly as

$$\lambda_{p,2} = \int_0^1 (1-u)^{2/p'} \mathrm{d}u = \frac{p}{3p-2}.$$

Proof. Existence and uniqueness of a minimizer of (16), follows from a classical argument using the direct method and the fact that the energy $\mathcal{C}_{p,d}$ is strictly convex. In addition, as this energy is invariant with respect to rotations around the origin, the solution must be radially symmetric and given by

$$u(x) = f_p(|x|).$$

From Morrey's inequality

$$[u]_{C^{0,\beta}(\mathbb{R}^{d-1})} \leq C_d \left\|\nabla u\right\|_{L^p(\mathbb{R}^{d-1})}, \ \text{for } \beta = 1 - \frac{d}{p},$$

we conclude that f_p must be β-Hölder continuous.

Let us show that f_p is the unique minimizer of (17). Given any u admissible for (16), from the coarea formula [24, Thm. 3.13] and a change of variables, it holds that

$$\begin{aligned} \mathcal{C}_{p,d}(u) &= \int_0^\infty \left(\int_{t\mathbb{S}^{d-2}} \frac{1}{p}|\nabla u|^p + \frac{1}{p'}(1-u)^2 \mathrm{d}\mathscr{H}^{d-2}\right) \mathrm{d}t \\ &= \int_{\mathbb{S}^{d-2}} \int_0^\infty t^{d-2}\left(\frac{1}{p}|\nabla u(t\xi)|^p + \frac{1}{p'}(1-u(t\xi))^2 \mathrm{d}t\right) \mathrm{d}\mathscr{H}^{d-2}(\xi) \\ &\geq \int_{\mathbb{S}^{d-2}} \int_0^\infty t^{d-2}\left(\frac{1}{p}|(u(t\xi))'|^p + \frac{1}{p'}(1-u(t\xi))^2 \mathrm{d}t\right) \mathrm{d}\mathscr{H}^{d-2}(\xi) \geq \sigma_{d-2}\lambda_{p,d}, \end{aligned}$$

where the last inequality comes from the fact that, for all $\xi \in \mathbb{S}^{d-2}$, the function $t \mapsto u(t\xi)$ is admissible for the problem (17), as $1-u \in L^2(\mathbb{R}^{d-1})$. As a result, if f_p is not the unique solution to (17), then there is another function $\bar{f}$ such that

$$\lambda_{p,d} < c_{p,d}(\bar{f}) < c_{p,d}(f_p),$$

so that $\bar{u}(x) = \bar{f}(|x|)$ has a strictly smaller energy that $u(x) = f_p(|x|)$. Hence, f_p must be a minimizer of (17), which must be unique from the uniqueness of solutions to (16). As a consequence, we conclude that $\Lambda_{p,d} = \sigma_{d-2}\lambda_{p,d}$.

The fact that f_p has image in $[0,1]$ comes from the fact that, if it was not the case, we could replace it with $\max\{0, \min\{f_p, 1\}\}$ and obtain a strictly smaller energy. Similarly, the strict monotonicity of f_p is achieved by replacing f_p with

$$\bar{f}_p(t) \stackrel{\text{def.}}{=} \max\{f_p(s) :\ 0 \le s \le t\},$$

yielding a strictly better energy if f_p and $\bar{f}_p$ do not coincide.

The variational problem (17) can become quite intractable for general p and d, but for the special case $d = 2$, we can refine the previous argument since the Lagrangian for (17) is now autonomous. In this case, notice that for any f admissible, we obtain a lower bound for $\lambda_{p,2}$ as

$$\begin{aligned}\int_0^{+\infty} \left(\frac{1}{p}|f'(t)|^p + \frac{1}{p'}(1-f(t))^2\right) \mathrm{d}t &\ge \int_0^{+\infty} (1-f(t))^{2/p'}|f'(t)|\mathrm{d}t \\ &= \lim_{t\to\infty} \int_0^{f(t)} (1-u)^{2/p'}\mathrm{d}u = \int_0^1 (1-u)^{2/p'}\mathrm{d}u\end{aligned}$$

where the inequality comes from Young's inequality, $ab \le \frac{1}{p}a^p + \frac{1}{p'}b^{p'}$ which is an equality if and only if $a = b^{p'-1}$, or equivalently $b = a^{p-1}$. An optimal solution f_p to the 1D variational problem must then satisfy Young's inequality with equality for a.e. t, and hence it must solve the ODE

$$f_p'(t) = (1-f_p(t))^{2/p}, \quad f_p(0) = 0. \tag{18}$$

It is then straightforward to verify that f_p is given by

$$f_p(t) = 1 - \left(1 - \frac{p-2}{p}t\right)^{\frac{p}{p-2}}, \text{ for } t \in \left[0, \frac{p}{p-2}\right],$$

and for $t > p/(p-2)$ we extend f_p with 1 and the integral remains unchanged. Using f_p in the energy from (17), we attain the lower bound for $\lambda_{p,2}$ above, and it follows that f_p must be the unique minimizer. ☐

An analogous analysis can be performed for the case $p = d = 2$. In this case, the same argument with Young's inequality gives the ODE $f'(t) = 1 - f(t)$ with the same boundary conditions, whose solution is given by $f_2(t) = 1 - e^{-t}$, for $t \ge 0$. Now the optimal profile never attains the value 1 and is more diffuse, see Figure (3.1). Compare it also with the proof of Lemma 2.8 in [10].

Remark 3.2 (On the regularity and the support of f_p)**.** In the case $d > 2$, it is not clear if f_p attains the value 1 in finite time. This is the case for $d = 2$, and should also hold for $d > 2$, as the extra term t^{d-2} penalizes even more f_p being away from 1.

In addition, besides being globally Hölder continuous, from the Euler-Lagrange equations, it follows that f_p is C^1 with a Hölder continuous derivative.

In the sequel, given a connected and countably $\mathscr{H}^1$-rectifiable set Σ, we use this optimal profile to construct a family of phase-fields $(\varphi_\varepsilon)_{\varepsilon>0}$ such that the associated diffuse transition measures μ_ε approximate $\mathscr{H}^1 \llcorner \Sigma$. Our strategy will be to combine the optimal profile obtained in Prop. 3.1 with the fact that the Minkowski content coincides with

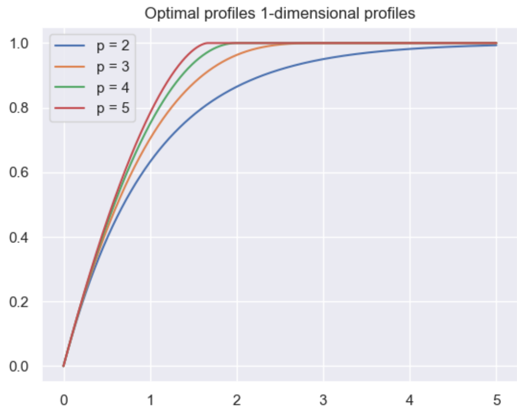

FIGURE 1. Optimal profiles induced by the Ambrosio-Tortorelli functional in $\mathbb{R}^2$ with different values of p. From the behavior of the optimal solution as p grows, one can expect that phase-field approximations with $p > 2$ yields sharper results.

the Hausdorff measure, see [4, Thm. 2.104]. More precisely, if Σ is closed and countably $\mathscr{H}^1$-rectifiable, defining $\Sigma_t \overset{\text{def.}}{=} \{x : \operatorname{dist}(x, \Sigma) \leq t\}$ it holds that

$$\lim_{t \to 0} \frac{\mathcal{L}^d(\Sigma_t)}{\omega_{d-1} t^{d-1}} = \mathscr{H}^1(\Sigma), \tag{19}$$

where ω_{d-1} denotes the volume of the unitary $d-1$-dimensional ball. This property is not always true; if Σ is a rectifiable curve it is known to be true, see [26, Thm. 3.2.39]. Alternatively, the conclusion (19) also holds if there is a Radon measure μ over Σ that is Ahlfors regular from below, [4, Thm. 2.104]; that is, there exists a constant $c > 0$ and some $r_0 > 0$ such that for all $x \in \Sigma$ if holds that

$$\mu(B_r(x)) \geq cr, \text{ for all } x \in \Sigma \text{ and } r < r_0.$$

Of course this is true for any path-connected set Σ by taking $\mu = \mathscr{H}^1 \llcorner \Sigma$ since for any $r < \operatorname{diam}(\Sigma)/2$ and $x \in \Sigma$ we have $\mathscr{H}^1(\Sigma \cap B_r(x)) \geq r$, so for any set we might be interested in this work, its Hausdorff measure coincides with the Minkowski content. We shall use in fact that this equality implies a weak convergence in the space of measures.

Lemma 3.3. *Let Σ be a compact, connected and countably $\mathscr{H}^1$-rectifiable subset of $\mathbb{R}^d$ with finite length $\mathscr{H}^1(\Sigma) < \infty$, then it holds that*

$$\frac{1}{\omega_{d-1} t^{d-1}} \mathcal{L}^d \llcorner \Sigma_t \xrightarrow[t \to 0]{\star} \mathscr{H}^1 \llcorner \Sigma.$$

Proof. Set $\nu_t \overset{\text{def.}}{=} \frac{1}{\omega_{d-1} t^{d-1}} \mathcal{L}^d \llcorner \Sigma_t$ for $t > 0$, notice that property (19) implies that $\nu_t(\mathbb{R}^d) \xrightarrow[t \to 0]{} \mathscr{H}^1(\Sigma)$. Let ν be a weak cluster point of ν_t, if we show that $\nu \geq \mathscr{H}^1 \llcorner \Sigma$, the convergence of the total mass implies that $\nu = \mathscr{H}^1 \llcorner \Sigma$.

From [4, Prop 2.101], which shows that the "lower Minkowski content" of a rectifiable set is larger than its Hausdorff measure, as the set $\Sigma \cap \overline{B_r(x)}$ is closed and countably $\mathscr{H}^1$-rectifiable, for any $x \in \mathbb{R}^d$ and $0 < r' < r$ it holds that

$$\begin{aligned} \nu(B_r(x)) &\geq \limsup_{t \to 0} \nu_t\left(\overline{B_{r'}(x)}\right) \geq \liminf_{t \to 0} \frac{\mathcal{L}^d\left(\left\{\operatorname{dist}(\cdot, \Sigma \cap \overline{B_{r'}(x)}) \leq t\right\}\right)}{\omega_{d-1} t^{d-1}} \\ &\geq \mathscr{H}^1\left(\Sigma \cap \overline{B_{r'}(x)}\right), \text{ for } r' < r. \end{aligned}$$

Letting $r' \to r$, we conclude that $\nu \geq \mathscr{H}^1 \llcorner \Sigma$, and equality follows since both measures have the same total mass. Since all cluster points of ν_t are $\mathscr{H}^1 \llcorner \Sigma$, it must be the weak limit of the entire family. $\square$

Now we prove the promised approximation result in $\mathbb{R}^d$, which is a strengthened version of [10, Lemma 2.8], from where the main idea of the proof is borrowed. In [10] the corresponding result is not stated as a weak convergence of the diffuse transition measure and is only proved in $\mathbb{R}^2$. Although the weak convergence is a small improvement to [10], it is crucial to the proof of the Γ-convergence result for $(\overline{W\mathscr{H}^1})$. We expect that the diffuse transition measures μ_ε associated with the family

$$\varphi_\varepsilon(x) \stackrel{\text{def.}}{=} \begin{cases} f_p\left(\dfrac{d_\Sigma(x) - b_\varepsilon}{\varepsilon}\right), & \text{if } d_\Sigma(x) \geq b_\varepsilon \\ 0, & \text{otherwise,} \end{cases} \tag{20}$$

will converge to $\mathscr{H}^1 \llcorner \Sigma$, where $d_\Sigma(\cdot) \stackrel{\text{def.}}{=} \operatorname{dist}(\cdot, \Sigma)$ and $b_\varepsilon = o(\varepsilon)$. We only need to be careful with the boundary condition $\varphi_\varepsilon \in 1 + W_0^{1,p}(\Omega)$, this is the case if f_p reaches 1 in finite time, for instance if $p > d = 2$.

Theorem 3.4 (Approximation with diffuse measures)**.** *Given $\Sigma \subset \Omega$ closed, connected with $\mathscr{H}^1(\Sigma) < \infty$. Then, there is a family $(\varphi_\varepsilon)_{\varepsilon>0} \subset 1 + W_0^{1,p}(\Omega)$ whose corresponding diffuse approximation measures defined in* (7) *are such that*

$$\mu_\varepsilon \xrightarrow[\varepsilon\to 0]{} \mathscr{H}^1 \llcorner \Sigma,$$

in both the narrow topology and the weak-$\star$ topologies of $\mathcal{M}_b(\Omega)$.

If $\Sigma \subset \operatorname{int}\Omega$, this family $(\varphi_\varepsilon)_{\varepsilon>0}$ can be constructed such that $\varphi_\varepsilon \equiv 0$ over the set $\{\operatorname{dist}(\cdot, \Sigma) \leq b_\varepsilon\}$, for $b_\varepsilon = o(\varepsilon)$.

Proof. The proof will be done in multiple cases of increasing generality, depending if Σ has a part contained in the boundary of Ω and if the optimal profile f_p reaches 1 in finite time. As a preliminary result, we prove an approximation result in the entire space $\mathbb{R}^d$. Define $d_\Sigma(x) \stackrel{\text{def.}}{=} \operatorname{dist}(x, \Sigma)$, set the notation

$$\Sigma_r \stackrel{\text{def.}}{=} \{x \in \Omega : d_\Sigma(x) \leq r\}$$

and let $(\varphi_\varepsilon)_{\varepsilon>0}$ be the family defined in (20). Consider now the measures over $\mathcal{M}_b(\mathbb{R}^d)$

$$\varrho_\varepsilon \stackrel{\text{def.}}{=} \frac{1}{\Lambda_{p,d}}\left(\frac{\varepsilon^{p-d+1}}{p}|\nabla\varphi_\varepsilon|^p + \frac{\varepsilon^{-d+1}}{p'}(1-\varphi_\varepsilon)^2\right)\mathcal{L}^d. \tag{21}$$

To show the convergence of $(\varrho_\varepsilon)_{\varepsilon>0}$ to $\mathscr{H}^1 \llcorner \Sigma$, our strategy will be to use the Minkowski content of Σ, and more specifically Lemma 3.3, to verify

$$\int_{\mathbb{R}^d} \psi \mathrm{d}\varrho_\varepsilon \xrightarrow[\varepsilon\to 0]{} \int_\Sigma \psi \mathrm{d}\mathscr{H}^1 \text{ for any } \psi \in C_b(\mathbb{R}^d). \tag{22}$$

Fixing $\psi \in C_b(\mathbb{R}^d)$, from the coarea formula [24, Thm. 3.13], we can define

$$\Psi : t \mapsto \int_{\Sigma_t} \psi \mathrm{d}x, \text{ such that } \Psi'(t) = \int_{\partial\Sigma_t} \psi \mathrm{d}\mathscr{H}^{d-1} \text{ for a.e. } t > 0.$$

It follows that Ψ is bounded, and Lemma 3.3 implies that

$$\Psi(t) = \omega_{d-1} t^{d-1} \int_\Sigma \psi \mathrm{d}\mathscr{H}^1 + o(t^{d-1}), \tag{23}$$

where the $o(t^{d-1})$ depends only on Σ and the function ψ.

Since the functions φ_ε are defined as the composition of a 1 dimensional profile with the distance function d_Σ, we can disintegrate then with the sets $\partial\Sigma_t$, over which φ_ε is constant. For this reason, we define the quantity

$$\begin{aligned} h_\varepsilon(t) &\overset{\text{def.}}{=} \frac{\varepsilon^{p-d+1}}{p}\left(\frac{\mathrm{d}}{\mathrm{d}t} f_p\left(\frac{t}{\varepsilon}\right)\right)^p + \frac{\varepsilon^{-d+1}}{p'}\left(1 - f_p\left(\frac{t}{\varepsilon}\right)\right)^2 \\ &= \varepsilon^{-d+1}\left(\frac{1}{p}\left|f_p'\left(\frac{t}{\varepsilon}\right)\right|^p + \frac{1}{p'}\left(1 - f_p\left(\frac{t}{\varepsilon}\right)\right)^2\right). \end{aligned}$$

Notice that since f_p is the optimal 1-dimensional profile from the problem (17), it follows that $h_\varepsilon(t) \xrightarrow[t\to\infty]{} 0$, for all $\varepsilon > 0$.

In the sequel we decompose the integral in (22) as

$$\int_{\mathbb{R}^d} \psi \mathrm{d}\varrho_\varepsilon = \int_{\Sigma_{b_\varepsilon}} \psi \mathrm{d}\varrho_\varepsilon + \int_{\mathbb{R}^d\setminus\Sigma_{b_\varepsilon}} \psi \mathrm{d}\varrho_\varepsilon$$

Since $\varphi_\varepsilon \equiv 0$ over Σ_{b_ε}, the first integral on the right-hand side above becomes

$$\int_{\Sigma_{b_\varepsilon}} \psi \mathrm{d}\varrho_\varepsilon = \frac{\varepsilon^{-d+1}}{p'\Lambda_{p,d}}\Psi(b_\varepsilon) = \left(\frac{\omega_{d-1}}{p'\Lambda_{p,d}}\int_\Sigma \psi \mathrm{d}\mathscr{H}^1\right)\left(\frac{b_\varepsilon}{\varepsilon}\right)^{d-1} + \frac{o({b_\varepsilon}^{d-1})}{\varepsilon^{d-1}} \xrightarrow[\varepsilon\to 0]{} 0,$$

which converges to 0 since $b_\varepsilon = o(\varepsilon)$.

Hence, in order to study the convergence (22), it suffices to consider the second term, which can be rewritten with the coarea formula as

$$\begin{aligned} \int_{\mathbb{R}^d\setminus\Sigma_{b_\varepsilon}} \psi \mathrm{d}\varrho_\varepsilon &= \frac{1}{\Lambda_{p,d}}\int_0^{+\infty} h_\varepsilon(t)\Psi'(t+b_\varepsilon)\mathrm{d}t \\ &= \frac{1}{\Lambda_{p,d}}\left(h_\varepsilon(t)\Psi(t+b_\varepsilon)\big|_0^{+\infty} - \int_0^\infty h_\varepsilon'(t)\Psi(t+b_\varepsilon)\mathrm{d}t\right) \end{aligned}$$

Recalling that $h_\varepsilon(t) \xrightarrow[t\to\infty]{} 0$ and that Ψ is a bounded function such that $\Psi(b_\varepsilon) = o(\varepsilon^{d-1})$, from the Minkowski content, the boundary terms vanish at the limit and we retain once again just the integral part, which we develop further as

$$\begin{aligned} \left(-\int_0^\infty h_\varepsilon'(t)\Psi(t+b_\varepsilon)\mathrm{d}t\right) &= \omega_{d-1}\left(\int_\Sigma \psi \mathrm{d}\mathscr{H}^1 + o(1)\right)\int_0^\infty -(t+b_\varepsilon)^{d-1}h_\varepsilon'(t)\mathrm{d}t \\ &= \omega_{d-1}(d-1)\left(\int_\Sigma \psi \mathrm{d}\mathscr{H}^1 + o(1)\right)\int_0^\infty (t+b_\varepsilon)^{d-2}h_\varepsilon(t)\mathrm{d}t, \end{aligned}$$

where we have used (23) in the first equality. Using the fact that h_ε is obtained with the optimal profile defining the constant $\lambda_{p,d}$ we obtain

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} \psi \mathrm{d}\varrho_\varepsilon = \frac{\omega_{d-1}(d-1)}{\Lambda_{p,d}}\left(\int_\Sigma \psi \mathrm{d}\mathscr{H}^1\right)\lim_{\varepsilon\to 0}\underbrace{\int_0^\infty \left(t + \frac{b_\varepsilon}{\varepsilon}\right)^{d-2}\left(\frac{1}{p}\left|f_p'\right|^p + \frac{1}{p'}(1-f_p)^2\right)\mathrm{d}t}_{=:\lambda_{p,d,\varepsilon}}$$

From Lebesgue's dominated convergence theorem, $\lambda_{p,d,\varepsilon} \xrightarrow[\varepsilon\to 0]{} \lambda_{p,d}$, since $b_\varepsilon = o(\varepsilon)$. Hence, recalling that $\omega_{d-1}(d-1) = \sigma_{d-2}$ and that, $\Lambda_{p,d} = \sigma_{d-2}\lambda_{p,d}$ from Proposition 3.1, we obtain the desired convergence

$$\int_{\mathbb{R}^d} \psi \mathrm{d}\varrho_\varepsilon \xrightarrow[\varepsilon\to 0]{} \int_\Sigma \psi \mathrm{d}\mathscr{H}^1$$

for all $\psi \in C_b(\mathbb{R}^d)$, so that $\varrho_\varepsilon \xrightarrow[\varepsilon\to 0]{} \mathscr{H}^1 \llcorner \Sigma$ in both the narrow and the weak-$\star$ topologies.

Now let us make the construction of approximating phase-fields with the additional constraint that $\varphi_\varepsilon \in 1 + W_0^{1,p}(\Omega)$. As mentioned in the beginning, we shall divide our construction in different cases.

Case 1 ($\Sigma \subset \operatorname{int}\Omega$ and $\operatorname{supp}(1 - f_p)$ is compact):

Given $\Sigma \subset \operatorname{int}\Omega$, since $1-f_p$ has compact support, the family $(\varphi_\varepsilon)_{\varepsilon>0}$ defined in (20) is contained in $1+W_0^{1,p}(\Omega)$. Indeed, setting $t_p \overset{\text{def.}}{=} \inf\{t \geq 0 : f_p(t)=1\} < \infty$, we get that $\varphi_\varepsilon < 1$ only when

$$\frac{d_\Sigma(\cdot)-b_\varepsilon}{\varepsilon} \leq t_p, \text{ so that } \operatorname{supp}(1-\varphi_\varepsilon) \subset \Sigma_{\varepsilon t_p + b_\varepsilon} \subset \operatorname{int}\Omega,$$

whenever $\varepsilon t_p + b_\varepsilon < \operatorname{dist}(\Sigma, \partial\Omega)$. Therefore, for ε small enough, we have $\mu_\varepsilon = \varrho_\varepsilon$ and the result follows.

Case 2 ($\Sigma \subset \operatorname{int}\Omega$ and $\operatorname{supp}(1-f_p)$ not compact):

When we can no longer assume that the support of $1-f_p$ is compact, we approximate it with another profile with compact support. In this case, set

$$t_\varepsilon \overset{\text{def.}}{=} \frac{\operatorname{dist}(\Sigma, \partial\Omega)}{2\varepsilon}, \quad \lambda_\varepsilon \overset{\text{def.}}{=} f_p(t_\varepsilon) \xrightarrow[\varepsilon\to 0]{} 1,$$

and define the new profiles

$$f_{p,\varepsilon}(t) \overset{\text{def.}}{=} \begin{cases} \dfrac{1}{\lambda_\varepsilon} f_p(t), & \text{if } t \leq t_\varepsilon, \\ 1, & \text{otherwise,} \end{cases} \qquad \bar{\varphi}_\varepsilon(x) \overset{\text{def.}}{=} \begin{cases} f_{p,\varepsilon}\left(\dfrac{d_\Sigma(x)-b_\varepsilon}{\varepsilon}\right), & \text{if } d_\Sigma(x) \geq b_\varepsilon \\ 0, & \text{otherwise.} \end{cases}$$

Similarly to the previous case, $\bar{\varphi}_\varepsilon \in 1+W_0^{1,p}(\Omega)$ for ε small enough.

Now let μ_ε denote the diffuse approximation measures referent to $\bar{\varphi}_\varepsilon$. Since we know that ϱ_ε from (21) converge weakly to $\mathscr{H}^1 \llcorner \Sigma$, to obtain the same limit for μ_ε it suffices to show that

$$\|\mu_\varepsilon - \varrho_\varepsilon\|_{L^1(\mathbb{R}^d)} \xrightarrow[\varepsilon\to 0]{} 0.$$

Indeed, a similar computation to the start of the proof using the coarea formula gives

$$\begin{aligned}
\|\mu_\varepsilon - \varrho_\varepsilon\|_{L^1(\mathbb{R}^d)} \leq & \int_{\mathbb{R}^d} \left(\frac{\varepsilon^{p-d+1}}{p}\left(\frac{1}{\lambda_\varepsilon^p}-1\right)|\nabla\varphi_\varepsilon|^p + \frac{\varepsilon^{-d+1}}{p}\left((1-\varphi_\varepsilon)^2 - \left(1-\frac{1}{\lambda_\varepsilon}\varphi_\varepsilon\right)^2\right)\right) \mathrm{d}x \\
=&\varepsilon^{-d+1}\int_{b_\varepsilon}^\infty \mathscr{H}^1(\Sigma_t)\left[\left(\frac{1}{\lambda_\varepsilon^p}-1\right)\frac{1}{p}\left|f_p'\left(\frac{t}{\varepsilon}\right)\right|^p \right.\\
&\left. + \frac{1}{p'}\left(\left(1-f_p\left(\frac{t}{\varepsilon}\right)\right)^2 - \left(1-\frac{1}{\lambda_\varepsilon}f_p\left(\frac{t}{\varepsilon}\right)\right)^2\right)\right]\mathrm{d}t \\
\leq & C\int_{b_\varepsilon/\varepsilon}^\infty s^{d-2}\left[\left(\frac{1}{\lambda_\varepsilon^p}-1\right)\frac{1}{p}\left|f_p'\right|^p + \frac{1}{p'}\left((1-f_p)^2 - \left(1-\frac{1}{\lambda_\varepsilon}f_p\right)^2\right)\right]\mathrm{d}s.
\end{aligned}$$

From Lebesgue's dominated convergence theorem, we conclude that $\|\mu_\varepsilon - \varrho_\varepsilon\|_{L^1(\mathbb{R}^d)} \xrightarrow[\varepsilon\to 0]{} 0$ so that $\mu_\varepsilon \xrightarrow[\varepsilon\to 0]{} \mathscr{H}^1 \llcorner \Sigma$.

Case 3 ($\Sigma \subset \Omega$ and $\operatorname{supp}(1-f_p)$ not compact): For this case we exploit the assumption that Ω is star-shaped to define a sequence of sets $\Sigma_n \subset \operatorname{int}\Omega$ and such that $\mathscr{H}^1 \llcorner \Sigma_n \xrightarrow[N\to\infty]{} \mathscr{H}^1 \llcorner \Sigma$. Notice that we make this assumption to slightly simplify the proof, but a similar construction can be made by assuming that Ω has a continuous boundary any using a partition of the unity over the boundary.

We consider $x_\star \in \operatorname{int}\Omega$ such that $tx_\star + (1-t)x \in \operatorname{int}\Omega$ for any $x \in \Omega$ and any $t \in (0,1)$.

So considering the sequence

$$\Sigma_n \overset{\text{def.}}{=} \frac{1}{n}x_\star + \left(1-\frac{1}{n}\Sigma\right), \text{ if holds } \mathscr{H}^1 \llcorner \Sigma_n \xrightarrow[\varepsilon\to 0]{} \mathscr{H}^1 \llcorner \Sigma.$$

We let $(\varphi_{n,\varepsilon})_{\varepsilon>0}$ be the family in $1+W_0^{1,p}(\Omega)$ obtained in the previous case whose diffuse approximation measures $(\mu_{n,\varepsilon})_{\varepsilon>0}$ are such that

$$\mu_{n,\varepsilon} \xrightarrow[\varepsilon\to 0]{} \mathscr{H}^1 \llcorner \Sigma_n \xrightarrow[n\to\infty]{} \mathscr{H}^1 \llcorner \Sigma.$$

Hence, a diagonal extraction argument yields the desired sequence. □

Notice that this proof also works for the case $p = 2$, using the corresponding optimal 1-dimensional profile, see Figure 3.1, as done in [10]. As discussed after Prop. 3.1, since the 1-dimensional profile for $p > 2$ promotes a sharper transition, the optimal sequence of phase fields constructed in the previous Theorem should have a better perceptual reconstruction. This is corroborated in Figure 3.1.

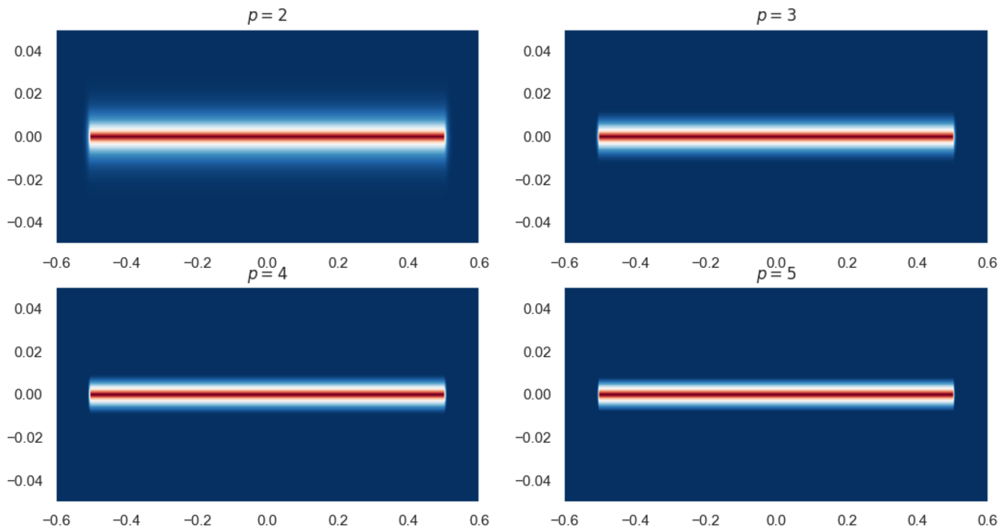

FIGURE 2. Recovery sequences (for $b_\varepsilon = 0$) obtained with the optimal profile from Prop. 3.1 for different values of p and $\varepsilon = 0.01$.

Remark 3.5. In Theorem 3.4, we have actually shown that the sequence of diffuse transition measures corresponding to the family defined in (20) is such that

$$\mu_\varepsilon \xrightarrow[\varepsilon \to 0]{} \mathscr{H}^1 \llcorner \Sigma, \text{ in } \mathcal{M}_b(\mathbb{R}^d).$$

If we had simply restricted φ_ε from (20) to Ω it would follow that

$$\mu_\varepsilon \xrightarrow[\varepsilon \to 0]{} \mathscr{H}^1 \llcorner (\Sigma \cap \operatorname{int} \Omega) + \frac{1}{2} \mathscr{H}^1 \llcorner (\Sigma \cap \partial\Omega), \text{ in } \mathcal{M}_b(\Omega).$$

3.2. **Properties of $\mathcal{C}_\varepsilon$.** In the sequel, we survey some properties of the connectedness functional. For a fixed parameter $s > 0$, the functional $\mathcal{C}_\varepsilon$ is designed to penalize the non-connectedness of the set $\{\varphi \leq \varepsilon^s\}$. Given a function $\Phi : \Omega \to [0, 1]$ we define the weighted distance d_ε^Φ as

$$d_\varepsilon^\Phi(x, y) \stackrel{\text{def.}}{=} \inf \left\{ \int_K \Phi(x) \mathrm{d}\mathscr{H}^1(x) : \begin{array}{c} K \text{ connected}, x, y \in K \subset \Omega \\ \mathscr{H}^1(K) \leq \omega(\varepsilon) \end{array} \right\}, \tag{24}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, monotone increasing function such that $\omega(\varepsilon) \to \infty$ as $\varepsilon \to 0$. This quantity can only be a distance if $\Phi > 0$, except for a set of Hausdorff dimension strictly smaller than 1, but we shall commit the abuse of calling it a geodesic distance even if it is not necessarily the case.

However, there is no guarantee of being able to find a such K connecting x and y with a length smaller than $\omega(\varepsilon)$. For this reason, let $\operatorname{diam}_{\text{geo}}(\Omega)$ denote the diameter of Ω w.r.t. the geodesic distance inside Ω, which can be defined as

$$\operatorname{dist}_{\text{geo}}(x, y) \stackrel{\text{def.}}{=} \min \left\{ \mathscr{H}^1(\gamma) : x, y \in \gamma \text{ and } \gamma \subset \Omega \text{ is connected} \right\}.$$

As Ω is bounded and connected with Lipschitz boundary $\mathrm{diam}_{\mathrm{geo}}(\Omega) < \infty$ and for ε small enough so that $\omega(\varepsilon) > \mathrm{diam}_{\mathrm{geo}}(\Omega)$, there must be an admissible curve connecting x, y, so that the infimum (24) is bounded by $\|\Phi\|_\infty \, \mathrm{dist}_{\mathrm{geo}}(x, y)$.

Either way, assuming ε small enough, if we compose φ_ε with a function $F_\varepsilon(z)$ that is zero if $z \leq \varepsilon^s$, the quantity $d_\varepsilon^{F_\varepsilon \circ \varphi_\varepsilon}(x, y)$ gives a quantitative notion of how disconnected the two points x, y are in the set $\{\varphi_\varepsilon \leq \varepsilon^s\}$. To get a global notion, we must integrate among all pairs of points in this level set. This way, the *diffuse connectedness functional* $\mathcal{C}_\varepsilon$ from [19, 21] is then defined as

$$\mathcal{C}_\varepsilon(\varphi_\varepsilon) \stackrel{\text{def.}}{=} \int_{\Omega\times\Omega} \beta_\varepsilon(\varphi_\varepsilon(x))\beta_\varepsilon(\varphi_\varepsilon(y)) d_\varepsilon^{F_\varepsilon \circ \varphi_\varepsilon}(x, y)\mathrm{d}x\mathrm{d}y, \tag{25}$$

where β_ε, F_ε are continuous functions such that

$$\beta_\varepsilon(z) = \begin{cases} 1 & \text{if } z \leq \varepsilon^s, \\ 0 & \text{if } z \geq 2\varepsilon^s, \end{cases} \quad F_\varepsilon(z) = \begin{cases} 0 & \text{if } z \leq \varepsilon^s/2, \\ 1 & \text{if } z \geq \varepsilon^s. \end{cases} \tag{26}$$

In addition, make the following hypothesis on these functions that, as (H1), will be assumed without statement

(H2) $\beta_\varepsilon, F_\varepsilon$ are strictly monotone in the intervals $(\varepsilon^s, 2\varepsilon^s)$ and $(\frac{1}{2}\varepsilon^s, \varepsilon^s)$ respectively and
$$F_\varepsilon\left(\frac{3}{4}\varepsilon^s\right) \geq \frac{1}{2}.$$

Next, in order to prove existence of solutions to the approximate functionals used in the Γ convergence result, we show that $\mathcal{C}_\varepsilon$ is continuous for uniform convergence.

Lemma 3.6. *Let Ω be a compact, connect set with Lipschitz boundary, and ε small enough so that $\omega(\varepsilon) > \mathrm{diam}_{geo}(\Omega)$. For $\varepsilon > 0$ fixed, the following facts hold*

(1) If $\Phi \in C\left(\overline{\Omega}\right)$, for every pair $x, y \in \Omega$ there is an optimal set K attaining the geodesic distance $d_\varepsilon^\Phi(x, y)$ defined in (24). In addition, this set can be taken a curve.

(2) The geodesic distance $d_\varepsilon^\Phi(\cdot, \cdot)$ is Lipschitz continuous w.r.t. Φ for the uniform convergence, with Lipschitz constant given by $\omega(\varepsilon)$.

(3) The connectedness functional $\mathcal{C}_\varepsilon$ is continuous w.r.t. uniform convergence of continuous functions.

Proof. To prove (1), fix $x, y \in \Omega$ and Φ continuous. Consider a minimizing sequence of compact and connected sets K_n, with uniformly bounded length $\mathscr{H}^1(K_n) \leq \omega(\varepsilon)$ approximating the infimum in (24). From Blaschke's Theorem [4, Thm. 6.1], we can extract a subsequence (not relabelled) converging in the Hausdorff metric to a connected set K, which must also contain the points x, y.

Consider now the measures $\nu_n \stackrel{\text{def.}}{=} \mathscr{H}^1 \llcorner K_n$, from the uniform bound on the lengths of K_n, we obtain that $\nu_n(\overline{\Omega}) \leq \omega(\varepsilon)$. Hence, as we are in a compact set, Prokhorov's compactness theorem implies that ν_n has a weak cluster point ν. In addition, from Gołab's theorem [15, Thm. 2.2] we know that $\nu \geq \mathscr{H}^1 \llcorner K$ and the lower semi-continuity of the total variation norm w.r.t. weak convergence of measures gives that

$$\mathscr{H}^1(K) \leq \nu(\overline{\Omega}) \leq \liminf_{n\to\infty} \nu_n(\overline{\Omega}) = \liminf_{n\to\infty} \mathscr{H}^1(K_n) \leq \omega(\varepsilon),$$

so that K remains admissible for (24).

Finally, since K_n is a minimizing sequence and from the continuity of Φ we get

$$d_\varepsilon^\Phi(x, y) \leq \int_K \Phi \mathrm{d}\mathscr{H}^1 \leq \int_\Omega \Phi \mathrm{d}\nu = \lim_{n\to\infty} \int_{K_n} \Phi \mathrm{d}\mathscr{H}^1 = d_\varepsilon^\Phi(x, y),$$

so K attains the distance $d^\Phi(x, y)$. But as $\mathscr{H}^1(K) < \infty$ and it is connected, it must be pathwise connected and countably $\mathscr{H}^1$-rectifiable, so that it can be covered by countably many Lipschitz curves. But as $x, y \in K$, we can find a curve $\gamma \subset K$ whose end points are x, y. From the rectifiability of K, γ must be composed of countably many Lipschitz arcs.

To prove (2), consider two continuous functions Φ_1 and Φ_2 and let K_2 be optimal for the definition of $d_\varepsilon^{\Phi_2}(x,y)$, so that in particular $x, y \in K_2$ and $\mathscr{H}^1(K_2) \leq \omega(\varepsilon)$. It then holds that

$$\begin{aligned} d_\varepsilon^{\Phi_1}(x,y) &\leq \int_{K_2} \Phi_1 \mathrm{d}\mathscr{H}^1 = d_\varepsilon^{\Phi_2}(x,y) + \int_{K_2} (\Phi_1 - \Phi_2) \mathrm{d}\mathscr{H}^1 \\ &\leq d_\varepsilon^{\Phi_2}(x,y) + \omega(\varepsilon) \left\| \Phi_1 - \Phi_2 \right\|_\infty . \end{aligned}$$

Changing the roles of Φ_1 and Φ_2 the result follows.

Item (3) then becomes a consequence of the dominated convergence theorem: let $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ uniformly, so that $\beta_\varepsilon \circ \varphi_n \xrightarrow[n\to\infty]{} \beta_\varepsilon \circ \varphi$ uniformly. The sequence $\beta_\varepsilon \circ \varphi_n$ remains uniformly bounded in n and so does $d_\varepsilon^{F_\varepsilon \circ \varphi_n}(x,y)$. As Ω is a bounded set, the dominated convergence theorem yields

$$\begin{aligned} \mathcal{C}_\varepsilon(\varphi) &= \int_{\Omega\times\Omega} \beta_\varepsilon(\varphi(x))\beta_\varepsilon(\varphi(y)) d^{F_\varepsilon\circ\varphi}(x,y)\mathrm{d}x\mathrm{d}y \\ &= \int_{\Omega\times\Omega} \lim_{n\to\infty} \beta_\varepsilon(\varphi_n(x))\beta_\varepsilon(\varphi_n(y)) d^{F_\varepsilon\circ\varphi_n}(x,y)\mathrm{d}x\mathrm{d}y \\ &= \lim_{n\to\infty} \int_{\Omega\times\Omega} \beta_\varepsilon(\varphi_n(x))\beta_\varepsilon(\varphi_n(y)) d^{F_\varepsilon\circ\varphi_n}(x,y)\mathrm{d}x\mathrm{d}y = \lim_{n\to\infty} \mathcal{C}_\varepsilon(\varphi_n), \end{aligned}$$

proving the continuity of $\varphi \mapsto \mathcal{C}_\varepsilon(\varphi)$ w.r.t. uniform convergence. □

Remark 3.7. Without the constraint $\mathscr{H}^1(K) \leq \omega(\varepsilon)$ in the definition of $d_\varepsilon^\Phi(\cdot,\cdot)$ in (24), we believe it is possible to construct counter examples to the continuity property (2), for instance if we take $\Phi = \mathrm{dist}(\cdot, F)$, the distance function to a fractal set F as Koch's snowflake [25].

3.3. **The fundamental liminf and limsup inequalities.** In order to establish Γ-convergence results for problems ($W\mathscr{H}^1$) and (ADM) we will study the behavior of families of functions such that the following functional is uniformly bounded

$$F_\varepsilon(\varphi_\varepsilon, \nu_\varepsilon) \stackrel{\text{def.}}{=} \mathcal{AT}_p(\varphi_\varepsilon) + \frac{1}{\varepsilon^\kappa}\mathcal{C}_\varepsilon(\varphi_\varepsilon) + \frac{1}{\varepsilon^\ell}\int_\Omega \varphi_\varepsilon \mathrm{d}\nu_\varepsilon. \tag{27}$$

Our first result in this direction is a characterization of cluster points from families of functions such that $F_\varepsilon(\varphi_\varepsilon, \nu_\varepsilon) \leq C$ remains bounded for all $\varepsilon > 0$. We show this limit is supported in a connected, countably $\mathscr{H}^1$-rectifiable set whose length is bounded by the liminf of $F_\varepsilon(\varphi_\varepsilon, \nu_\varepsilon)$.

Theorem 3.8. *Let Ω be a compact, connected subset of $\mathbb{R}^d$ with Lipschitz boundary, and suppose that $\ell > s$ and $\kappa > (2d+1)\frac{(s+1)p-d+1}{p-d}$. For any family $(\varphi_\varepsilon, \nu_\varepsilon)_{\varepsilon>0}$ such that for every $\varepsilon > 0$ it holds that $\varphi_\varepsilon \in 1 + W_0^{1,p}(\Omega)$, $\nu_\varepsilon \in \mathscr{P}_{ac}(\Omega)$ and*

$$F_\varepsilon(\varphi_\varepsilon, \nu_\varepsilon) \leq C,$$

then it follows that:

- *(i) For ε small enough, there exists a connected component of $\{\varphi_\varepsilon \leq 2\varepsilon^s\}$, denoted by Σ_ε, that contains $\{\varphi_\varepsilon \leq \varepsilon^s/2\}$.*
- *(ii) Up to subsequences, Σ_ε converges in the Hausdorff distance to a connected countably $\mathscr{H}^1$-rectifiable set Σ and $\varphi_\varepsilon \xrightarrow[\varepsilon\to0]{} 1$, strongly in $L^2(\Omega)$. The families of measures $(\nu_\varepsilon, \mu_\varepsilon)_{\varepsilon>0}$, for μ_ε defined in (7), also converge $\nu_\varepsilon \xrightarrow[\varepsilon\to0]{\star} \nu$, $\mu_\varepsilon \xrightarrow[\varepsilon\to0]{\star} \mu$, and the limits satisfy*

$$\mu \geq \mathscr{H}^1 \llcorner \Sigma, \text{ and } \operatorname{supp}\nu \subset \Sigma \subset \operatorname{supp}\mu.$$

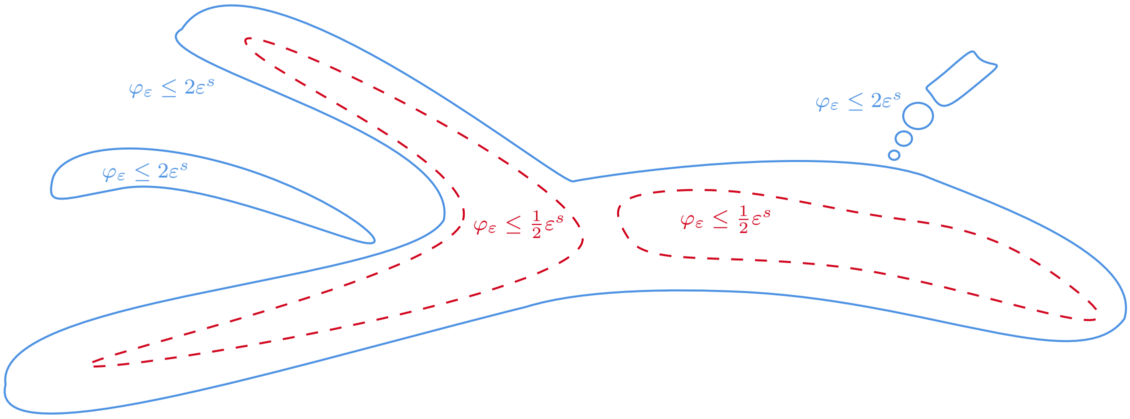

FIGURE 3. A single connected component of $\{\varphi_\varepsilon \leq 2\varepsilon^s\}$, Σ_ε, contains all of $\{\varphi_\varepsilon \leq \frac{1}{2}\varepsilon^s\}$ in red. Neither level sets are necessarily connected, but Σ_ε contains almost all the mass.

In particular, it holds that

$$\mathscr{H}^1(\Sigma) \leq \liminf_{\varepsilon\to 0} \frac{1}{\Lambda_{p,d}} \int_\Omega \left(\frac{\varepsilon^{p-d+1}}{p} |\nabla \varphi_\varepsilon|^p + \frac{\varepsilon^{-d+1}}{p'} (1-\varphi_\varepsilon)^2 \right) \mathrm{d}x. \tag{28}$$

Proof. **Step 1:** Our first step is to construct the connected and countably $\mathscr{H}^1$-rectifiable set Σ, where the cluster points of the diffuse measures μ_ε are concentrated. This will be done by studying the small level sets of the family $(\varphi_\varepsilon)_{\varepsilon>0}$. First we show that, for ε small enough, a single connected component of $\{\varphi_\varepsilon \leq 2\varepsilon^s\}$ contains all of $\{\varphi_\varepsilon \leq \varepsilon^s/2\}$, see Figure 3.3. Write

$$\{\varphi_\varepsilon < \varepsilon^s\} = \bigcup_{i\in I} \Sigma_{\varepsilon,i},$$

where $(\Sigma_{\varepsilon,i})_{i\in I}$ denote the set of connected components of $\{\varphi_\varepsilon < \varepsilon^s\}$. We distinguish the components that intersect the set we are interested in, $\{\varphi_\varepsilon < \varepsilon^s/2\}$, by defining the subset of indices

$$I_\star \overset{\text{def.}}{=} \{i \in I : \quad \Sigma_{\varepsilon,i} \cap \{\varphi_\varepsilon < \varepsilon^s/2\} \neq \emptyset\}.$$

Since $\varphi_\varepsilon \in 1+W_0^{1,p}(\Omega)$, any sublevel set $\{\varphi_\varepsilon \leq l\}$ is compactly contained in Ω for $0 \leq l < 1$. Hence, we can manipulate these sets without worrying about border effect.

First we check that $I_\star$ is not empty; if it were, we would have $\varphi_\varepsilon \geq \frac{1}{2}\varepsilon^s$ everywhere in Ω so that the term

$$\frac{\varepsilon^{s-\ell}}{2} |\nu_\varepsilon|(\Omega) \leq \frac{1}{\varepsilon^\ell} \int_\Omega \varphi_\varepsilon \mathrm{d}\nu_\varepsilon \leq C$$

would yield a contradiction letting $\varepsilon \to 0$ since we have assumed $\ell > s$.

We claim that there is a radius r_ε such that for all $i \in I_\star$ and any $x \in \Sigma_{\varepsilon,i} \cap \{\varphi_\varepsilon \leq \varepsilon^s/2\}$ or $x' \in \{\varphi_\varepsilon = \varepsilon^s\}$ one has that

$$B(x; r_\varepsilon) \subset \Sigma_{\varepsilon,i} \text{ and } B(x'; r_\varepsilon) \subset \left\{ \frac{3}{4}\varepsilon^s \leq \varphi_\varepsilon \leq 2\varepsilon^s \right\}. \tag{29}$$

This is a consequence of the fact that φ_ε is Hölder continuous. Indeed, since $p > d$, from Morrey's inequality (see [23, Thm. 5.4]) it holds that $\varphi_\varepsilon \in C^{0,\beta}$ with $\beta = 1 - \frac{d}{p}$ and a

Hölder constant bounded by

$$[\varphi_\varepsilon]_{C^{0,\beta}(B_r)} \le c \, \|\nabla \varphi_\varepsilon\|_{L^p(B_r)} \le c\varepsilon^{-\frac{p-d+1}{p}},$$

where the two constants above differ, the first depends only on the dimension of Ω and the second inequality follows from the bound on $F_\varepsilon(\varphi_\varepsilon)$. Hence, it follows directly from the definition of Hölder continuity that (29) holds with

$$r_\varepsilon = c_0 \varepsilon^{\beta'} \text{ with } \beta' > \frac{(s+1)p-d+1}{p-d}. \tag{30}$$

In particular, we conclude that $I_\star$ is finite since taking exactly one ball for each of these connected components we obtain the bound $|I_\star|\omega_d r_\varepsilon^d \le |\{\varphi_\varepsilon \le 2\varepsilon^s\}| < \infty$.

In the sequel we define the quantities

$$\delta_{ij} \overset{\text{def.}}{=} d_\varepsilon^{F_\varepsilon \circ \varphi_\varepsilon}(\Sigma_{\varepsilon,i}, \Sigma_{\varepsilon,j}) \overset{\text{def.}}{=} \min_{x \in \Sigma_{\varepsilon,i}, y \in \Sigma_{\varepsilon,j}} d_\varepsilon^{F_\varepsilon \circ \varphi_\varepsilon}(x,y), \quad \text{for } i \ne j \in I_\star.$$

In the sequel, using the balls of radius r_ε defined in (29), we can bound δ_{ij} from above and from below. Starting with the upper bound, by definition, it must hold that $\delta_{ij} \le d^{F_\varepsilon \circ \varphi_\varepsilon}(x,y)$ for any pair $x \in \Sigma_{\varepsilon,i}$ and $y \in \Sigma_{\varepsilon,j}$. Therefore, taking B_i, B_j as in (29) with centers in $\{\varphi_\varepsilon \le \varepsilon^s/2\}$, so that they are contained in $\{\varphi_\varepsilon \le \varepsilon^s\}$, we can bound the connectedness functional from below as

$$\left(\omega_d r_\varepsilon^d\right)^2 \delta_{ij} \le \int_{B_i \times B_j} \beta_\varepsilon(\varphi_\varepsilon(x))\beta_\varepsilon(\varphi_\varepsilon(y)) d^{F_\varepsilon \circ \varphi_\varepsilon}(x,y) \mathrm{d}x\mathrm{d}y \le \mathcal{C}_\varepsilon(\varphi_\varepsilon) \le C\varepsilon^\kappa,$$

where we have used the fact that $\beta_\varepsilon \circ \varphi_\varepsilon \equiv 1$ inside the set $\{\varphi_\varepsilon \le \varepsilon^s\}$, so that

$$\delta_{ij} \le C' \varepsilon^{\kappa - 2d\beta'}. \tag{31}$$

We conclude that, for ε small enough, $\delta_{ij} < r_\varepsilon$ for all $i \ne j \in I_\star$, since if it was not the case, we obtain a contradiction in

$$C'' \varepsilon^{\beta'} \le \delta_{ij} \le C' \varepsilon^{\kappa - 2d\beta'}, \text{ by taking } (2d+1)\beta' < \kappa,$$

so we chose $\frac{(2d+1)((s+1)-d+1)}{p-d} < \kappa$ and $\frac{(s+1)-d+1}{p-d} < \beta' < \frac{\kappa}{2d+1}$.

Now given any two $i, j \in I_\star$, letting γ be a curve attaining δ_{ij}. If $\Sigma_{\varepsilon,i}$ and $\Sigma_{\varepsilon,j}$ are not in the same connected component of $\{\varphi_\varepsilon \le 2\varepsilon^s\}$, then there are at least two points x_0, x_1 in this curve γ such that $\varphi_\varepsilon(x_0) = \varphi_\varepsilon(x_1) = \varepsilon^s$. From (29), there are balls $B(x_0, r_\varepsilon)$ and $B(x_1, r_\varepsilon)$, since each of then remains in the connected components' of $\{\varphi_\varepsilon \le 2\varepsilon^s\}$ that contain $\Sigma_{\varepsilon,i}$ and $\Sigma_{\varepsilon,j}$, respectively, so that $B(x_0, r_\varepsilon)$ and $B(x_1, r_\varepsilon)$ must be disjoint. This construction is illustrated in Figure 3.3.

Also from (29), both of these balls must be contained in $\{\frac{3}{4}\varepsilon^s \le \varphi_\varepsilon \le 2\varepsilon^s\}$ and, it holds from (H2) that $F_\varepsilon \circ \varphi_\varepsilon \ge \frac{1}{2}$ over $B(x_0, r_\varepsilon)$ and $B(x_1, r_\varepsilon)$. We then have that

$$\delta_{ij} \ge \int_{B(x_0,r_\varepsilon) \cup B(x_1,r_\varepsilon)} F_\varepsilon \circ \varphi_\varepsilon \mathrm{d}\mathscr{H}^1 \llcorner \gamma \ge \frac{1}{2} \mathscr{H}^1\left(\gamma \cap \left(B(x_0, r_\varepsilon) \cup B(x_1, r_\varepsilon)\right)\right) \ge r_\varepsilon.$$

As this is not true for ε sufficiently small, all $\Sigma_{\varepsilon,i}$ must be contained in the same connected component of $\{\varphi_\varepsilon \le 2\varepsilon^s\}$.

Let Σ_ε denote this connected component; up to subsequences, we can assume that $\Sigma_\varepsilon \xrightarrow[\varepsilon \to 0]{d_H} \Sigma$. As the Hausdorff limit of connected sets, Σ is itself connected. We can now show that if $\nu_\varepsilon \xrightarrow[\varepsilon \to 0]{\star} \nu$ then ν is concentrated in Σ. Since $\{\varphi_\varepsilon \le \frac{1}{2}\varepsilon^s\} \subset \Sigma_\varepsilon$, the energy bound $F_\varepsilon(\varphi_\varepsilon, \nu_\varepsilon) \le C$ implies

$$\nu_\varepsilon\left(\Omega \setminus \Sigma_\varepsilon\right) \le \frac{2}{\varepsilon^s} \int_{\Omega \setminus \Sigma_\varepsilon} \varphi_\varepsilon \mathrm{d}\nu_\varepsilon \le 2C\varepsilon^{\ell - s} \xrightarrow[\varepsilon \to 0]{} 0,$$

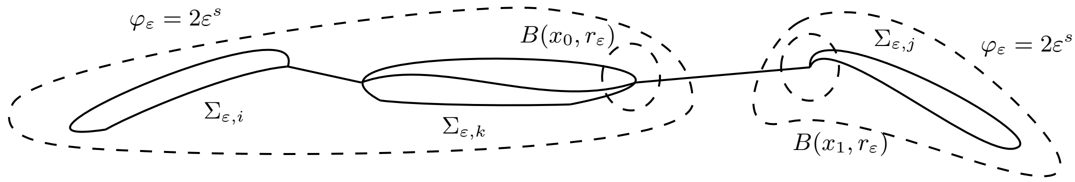

FIGURE 4. The optimal path between $\Sigma_{\varepsilon,i}$ and $\Sigma_{\varepsilon,j}$ has at least two segments of length r_ε.

as we have assumed that $\ell > s$. Therefore, if $x \notin \Sigma$, there is a radius r such that $B_r(x) \cap \Sigma_\varepsilon = \emptyset$ for all $\varepsilon > 0$ small enough. From the previous estimate and the properties of weak convergence, we obtain

$$\nu(B_r(x)) \leq \liminf_{\varepsilon \to 0} \nu_\varepsilon(B_r(x)) \leq \liminf_{\varepsilon \to 0} \nu_\varepsilon(\Omega \setminus \Sigma_\varepsilon) = 0.$$

This show that $\operatorname{supp} \nu \subset \Sigma$.

The rest of the proof is dedicated to show that any cluster point μ of μ_ε is such that $\mu \geq \mathscr{H}^1 \llcorner \Sigma$. We first show in Step 2 that $\mathscr{H}^1(\Sigma) < +\infty$, which will imply that Σ is countably $\mathscr{H}^1$-rectifiable. In Step 3 we use this fact to refine the estimates from Step 2 and conclude. Both of these arguments will be based on the fact that, see [4, Thm. 2.56],

$$\theta_1^\star(\mu, x) \overset{\text{def.}}{=} \limsup_{r \to 0} \frac{\mu\left(B_r(x)\right)}{2r} \geq \theta \text{ for } \mathscr{H}^1\text{-a.e. } x \in \Sigma \Longrightarrow \mu \geq \theta \mathscr{H}^1 \llcorner \Sigma. \tag{32}$$

Hence, in each of these Steps we prove an estimate of the form: for all $x \in \Sigma$

$$\liminf_{\varepsilon \to 0} \mu_\varepsilon(B_r(x)) \geq \theta 2r, \tag{33}$$

for different values of θ. As a consequence, this implies (32) by means of classical properties of weak convergence of measures.

Step 2: Given $x_0 \in \Sigma$, fix $r < \min\{\operatorname{dist}(x_0, \partial\Omega), \operatorname{diam}(\Sigma)/2\}$ so that $B_r(x_0) \subset \Omega$. Defining $v_\varepsilon = \varphi_\varepsilon(\varepsilon \cdot)$, we can rewrite $\mu_\varepsilon(B_r(x_0))$ as

$$\begin{aligned} \mu_\varepsilon(B_r(x_0)) &= \frac{\varepsilon^{-d+1}}{\Lambda_{p,d}} \int_0^r \left(\int_{\partial B_\rho(x_0)} \left[\frac{\varepsilon^p}{p} |\nabla \varphi_\varepsilon|^p + \frac{1}{p'}(1 - \varphi_\varepsilon)^2 \right] \mathrm{d}\mathscr{H}^{d-1} \right) \mathrm{d}\rho \\ &= \frac{1}{\Lambda_{p,d}} \int_0^r \left(\int_{\partial B\left(x_0, \frac{\rho}{\varepsilon}\right)} \left[\frac{1}{p} |\nabla v_\varepsilon|^p + \frac{1}{p'}(1 - v_\varepsilon)^2 \right] \mathrm{d}\mathscr{H}^{d-1} \right) \mathrm{d}\rho. \end{aligned} \tag{34}$$

From Fatou's Lemma we obtain that

$$\liminf_{\varepsilon \to 0} \mu_\varepsilon(B_r(x_0)) \geq \frac{1}{\Lambda_{p,d}} \int_0^r \left(\liminf_{\varepsilon \to 0} \int_{\partial B\left(x_0, \frac{\rho}{\varepsilon}\right)} \left[\frac{1}{p} |\nabla v_\varepsilon|^p + \frac{1}{p'}(1 - v_\varepsilon)^2 \right] \mathrm{d}\mathscr{H}^{d-1} \right) \mathrm{d}\rho. \tag{35}$$

Since the total mass of μ_ε is given by $\mathcal{AT}_p(\varphi_\varepsilon)$, the LHS above remains bounded and hence the liminf on the RHS of (35) is finite for a.e. $\rho \in (0, r)$. Hence, it suffices to bound this liminf from below with a constant that holds for almost every $\rho \in (0, r)$, in particular every ρ such that this liminf is finite suffices. To this end, our strategy will be to compare the inner integral in (35) with the auxiliary variational problem (16) that defines the constant $\Lambda_{p,d}$.

Our first step is to find some $x_\varepsilon \in \partial B_\rho(x_0)$ such that $\varphi_\varepsilon(x_\varepsilon) \leq 2\varepsilon^s$, for a fixed $\rho \in (0, r)$. We can assume that $\Sigma \setminus B_r(x_0)$ is not empty, so from the Hausdorff convergence of Σ_ε to Σ, for ε small enough, there is $z_\varepsilon \in B_\rho(x_0)$ and another point of Σ_ε outside B_r. But since Σ_ε is connected, there is some $x_\varepsilon \in \partial B_\rho(x_0) \cap \Sigma_\varepsilon$, with the desired property.

Up to a translation and a rotation, we may assume that $x_0 = -\rho e_d$ and $x_\varepsilon = 0$; and to define our new function over $\mathbb{R}^{d-1}$, first introduce the notation $\mathbb{R}^d \ni x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and define a map Φ_ε from the ball of $\mathbb{R}^{d-1}$, $B_{\mathbb{R}^{d-1}}(0, \frac{\rho}{\varepsilon})$, to the sphere $\partial B_{\mathbb{R}^d}(x_0, \frac{\rho}{\varepsilon})$ as

$$\Phi_\varepsilon(x') \overset{\text{def.}}{=} (x', \phi_\varepsilon(x')) \text{ where } \phi_\varepsilon(x') \overset{\text{def.}}{=} \sqrt{\left(\frac{\rho}{\varepsilon}\right)^2 - |x'|^2} - \frac{\rho}{\varepsilon}.$$

In the sequel, we obtain the new function $\tilde{v}_\varepsilon \in W^{1,p}(B_{\mathbb{R}^{d-1}}(0, \rho/2\varepsilon))$ as

$$\tilde{v}_\varepsilon(x') = v_\varepsilon(\Phi_\varepsilon(x')), \text{ for } x' \in B_{\mathbb{R}^{d-1}}(0, \rho/2\varepsilon).$$

Notice that $\nabla_{x'} \tilde{v}_\varepsilon = \nabla_{x'} \Phi_\varepsilon^\top \nabla_x v_\varepsilon \circ \Phi_\varepsilon$ so that

$$|\nabla_{x'} \tilde{v}_\varepsilon| = |\nabla_{x'} v_\varepsilon + \nabla_{x'} \phi_\varepsilon \partial_d v_\varepsilon| \leq C |\nabla_x v_\varepsilon|,$$

and using the area formula, one obtains that

(36)

$$\int_{B_{\mathbb{R}^{d-1}}(0, \frac{\rho}{2\varepsilon})} \left(\frac{1}{p}|\nabla \tilde{v}_\varepsilon|^p + \frac{1}{p'}(1 - \tilde{v}_\varepsilon)^2\right) \mathrm{d}x' \leq C \int_{\partial B(x_0, \frac{\rho}{\varepsilon})} \left(\frac{1}{p}|\nabla v_\varepsilon|^p + \frac{1}{p'}(1 - v_\varepsilon)^2\right) \mathrm{d}\mathscr{H}^{d-1},$$

for some constant $C > 0$ depending on the $(d-1)$-Jacobian of Φ_ε, more specifically the quantity

$$\det \left|\nabla_{x'} \Phi_\varepsilon^\top \nabla_{x'} \Phi_\varepsilon\right| = \left(1 + |\nabla_{x'} \phi_\varepsilon(x')|^2\right)^{1/2} = \left(1 + \frac{|\varepsilon x'|^2}{\rho^2 - |\varepsilon x'|^2}\right)^{1/2} = \begin{cases} \geq 1, \\ \leq \sqrt{3}, \end{cases}$$

which can be bounded from above and from below independently of ε for $x' \in B_{\mathbb{R}^{d-1}}(0, \frac{\rho}{2\varepsilon})$. Up to a subsequence, the left hand side of (36) is uniformly bounded since we have assumed the liminf in the right hand sind of (35) to be finite.

These estimates motivate the definition of a family of variational problems, indexed by ε, that approximate (16), the problem whose value defines the constant $\Lambda_{p,d}$, as follows

(37)

$$\Lambda_{p,d,\varepsilon} \overset{\text{def.}}{=} \min \left\{ \mathcal{C}_{p,d,\varepsilon}(u) \overset{\text{def.}}{=} \int_{B_{\mathbb{R}^{d-1}}(0, \frac{\rho}{2\varepsilon})} \left(\frac{1}{p}|\nabla w|^p + \frac{1}{p'}(1 - w)^2\right) \mathrm{d}x : \begin{array}{c} 1 - w \in L^2(\mathbb{R}^{d-1}), \\ \nabla w \in L^p(\mathbb{R}^{d-1}) \\ w(0) \leq 2\varepsilon^s \end{array} \right\},$$

and our goal is to show that $\Lambda_{p,d,\varepsilon} \xrightarrow[\varepsilon \to 0]{} \Lambda_{p,d}$. So let u be optimal for (16), then its restriction to $B_{\mathbb{R}^{d-1}}(0, \frac{\rho}{2\varepsilon})$ is admissible and we have

$$\Lambda_{p,d} \geq \int_{B_{\mathbb{R}^{d-1}}(0, \frac{\rho}{2\varepsilon})} \left(\frac{1}{p}|\nabla u|^p + \frac{1}{p'}(1 - u)^2\right) \mathrm{d}x \geq \Lambda_{p,d,\varepsilon}.$$

As a result, $\Lambda_{p,d,\varepsilon}$ is uniformly bounded and letting w_ε be a solution to (37), we will show that it converges locally uniformly to a function w. Indeed, for any $R > 0$ fixed, for ε large enough we have that $B_{\mathbb{R}^{d-1}(0,R)} \subset B_{\mathbb{R}^{d-1}(0, \frac{\rho}{2\varepsilon})}$, so the energy bound on $(w_\varepsilon)_{\varepsilon > 0}$ implies that this sequence is Hölder continuous, with the same constant, hence equicontinuous and equibounded from the fact that it converges to 1 at infinity. So, from Ascoli-Arzelà, this sequence converges to some w, uniformly in $B_{\mathbb{R}^{d-1}(0,R)}$. As this also implies the existence of a subsequence whose gradients converge weakly in L^p_{loc}, we get that

$$\liminf_{\varepsilon \to 0} \Lambda_{p,d,\varepsilon} \geq \int_{B_{\mathbb{R}^{d-1}(0,R)}} \left(\frac{1}{p}|\nabla w|^p + \frac{1}{p}(1 - w)^2\right) \mathrm{d}x.$$

Since w remains admissible for the problem defining $\Lambda_{p,d}$, taking the supremum on $R > 0$ we get that

$$\liminf_{\varepsilon \to 0} \Lambda_{p,d,\varepsilon} \geq \mathcal{C}_{p,d}(w) \geq \Lambda_{p,d}.$$

The desired convergence $\Lambda_{p,d,\varepsilon} \to \Lambda_{p,d}$ follows and we have that

$$C \liminf_{\varepsilon\to 0} \int_{\partial B\left(x_0, \frac{\rho}{2\varepsilon}\right)} \left[\frac{1}{p}|\nabla v_\varepsilon|^p + \frac{1}{p'}(1-v_\varepsilon)^2\right] \mathrm{d}\mathscr{H}^{d-1} \geq \liminf_{\varepsilon\to 0} \Lambda_{p,d,\varepsilon} = \Lambda_{p,d}.$$

Combining these estimates with (35), from the weak convergence of μ_ε to μ, we obtain

$$\begin{aligned}\mu\left(\overline{B_r(x_0)}\right) &\geq \liminf_{\varepsilon\to 0} \mu_\varepsilon(B_r(x_0)) \\ &\geq \frac{1}{C\Lambda_{p,d}} \int_0^r \left(\liminf_{\varepsilon\to 0} \int_{\partial B\left(x_0, \frac{\rho}{\varepsilon}\right)} \left[\frac{1}{p}|\nabla v_\varepsilon|^p + \frac{1}{p'}(1-v_\varepsilon)^2\right] \mathrm{d}\mathscr{H}^{d-1}\right) \mathrm{d}\rho \geq \theta 2r\end{aligned}$$

for $\theta = 1/(2C)$. So $\mu \geq \theta\mathscr{H}^1 \llcorner \Sigma$, from (32), and in particular $\mathscr{H}^1(\Sigma) < +\infty$.

Step 3: Now that we know that Σ has finite length, we deduce that it is rectifiable and we can use the rectifiability of Σ to refine the previous estimate, showing that $\theta_1^\star(\mu, x) \geq 1$ for $\mathscr{H}^1$-a.e. $x \in \Sigma$ and from (32) conclude that $\mu \geq \mathscr{H}^1 \llcorner \Sigma$ and (28) will follow from the properties of weak convergence of measures.

From the rectifiability of Σ it holds that $\mathscr{H}^1$-a.e. $x_0 \in \Sigma$ admits an approximate tangent space. Let x_0 be one of such points and assume, without loss of generality, that $T_{x_0}\Sigma = \mathbb{R}e_d$. So given a small radius r and $\delta \in (0,1)$ close to 1, we consider the cylinder

$$C_{\delta,r}(x_0) \overset{\text{def.}}{=} x_0 + \left\{x = (x', x_d) : \begin{array}{c}|x'| < \delta r \\ |x_d| < \delta' r\end{array}\right\} \text{ and } \delta' = \sqrt{1-\delta^2}.$$

Our goal is to refine the estimations from the previous step by taking a foliation given by planes orthogonal to $\mathbb{R}e_d$, instead of the spheres. In the sequel, for each t, we define a disc by slicing the cylinder $C_{\delta,r}(x_0)$ with the hyperplane $\{x_d = t\}$.

$$D_t \overset{\text{def.}}{=} C_{\delta,r}(x_0) \cap (\pi_d)^{-1}(x_0 + te_d), \text{ for } t \in (-\delta r, \delta r).$$

We can now obtain a more precise estimate than in Step 2. However, to obtain the point x_ε, such that $\varphi_\varepsilon(x_\varepsilon) \leq 2\varepsilon^s$, the connectedness of Σ_ε was sufficient since we could count on the spherical symmetry of ∂B_ρ. Now, we need a refined argument that will give a point $x_{\varepsilon,t} \in D_t$ such that $\varphi_\varepsilon(x_{\varepsilon,t}) \leq 2\varepsilon^s$, for almost every t. From item (3) of Theorem (2.1), we can find such point for all $t \in [-\delta r, \delta r] \setminus (a_\varepsilon, b_\varepsilon)$ with $b_\varepsilon - a_\varepsilon < 2d_H(\Sigma_\varepsilon, \Sigma)$. Now we can perform a computation analogous to the one presented in Step 2:

(38)
$$\begin{aligned}\mu_\varepsilon(B_r(x_0)) &\geq \mu_\varepsilon(C_{r,\delta}(x_0)) \\ &\geq \frac{1}{\Lambda_{p,d}} \int_{[-\delta r,\delta r]\setminus(a_\varepsilon,b_\varepsilon)} \left(\int_{D_t} \left(\frac{\varepsilon^{p-d+1}}{p}|\nabla\varphi_\varepsilon|^p + \frac{\varepsilon^{-d+1}}{p'}(1-\varphi_\varepsilon)^2\right) \mathrm{d}\mathscr{H}^{d-1}\right) \mathrm{d}t.\end{aligned}$$

Let us focus on the L^2 term for the moment. From the energy bound on $\mathcal{AT}_p(\varphi_\varepsilon)$, we know that

$$\int_{-\delta r}^{\delta r} \int_{D_t} (1-\varphi_\varepsilon)^2 \mathrm{d}\mathscr{H}^{d-1}\mathrm{d}t \leq C\varepsilon^{d-1}.$$

Hence, from the converse of the dominated convergence Theorem, up to a subsequence we can assume that, for a.e. $t \in (-\delta r, \delta r)$,

$$\int_{D_t} (1-\varphi_\varepsilon)^2 \mathrm{d}\mathscr{H}^{d-1} \xrightarrow[\varepsilon\to 0]{} 0.$$

Disintegrating once again, we can write the previous term as

$$\int_{D_t} (1-\varphi_\varepsilon)^2 \mathrm{d}\mathscr{H}^{d-1} = \int_{\mathbb{S}^{d-2}} \left(\int_0^{\delta' r} l^{d-2}(1-\varphi_\varepsilon(l\xi + te_d))^2 \mathrm{d}l\right) \mathrm{d}\mathscr{H}^{d-2}(\xi).$$

The same argument gives that, for a.e. $t \in (-\delta r, \delta r)$, for $\mathscr{H}^{d-2}$-a.e. $\xi \in \mathbb{S}^{d-2}$,

$$\varphi_\varepsilon(l\xi + te_d) \xrightarrow[\varepsilon \to 0]{} 1, \text{ for a.e. } l \in [0, \delta' r].$$

Now, fix $t \in (-\delta r, \delta r)$ and $\xi \in \mathbb{S}^{d-2}$ such that the previous limit holds and consider the point $x_{\varepsilon,t} \in D_t$ such that $\varphi_\varepsilon(x_{\varepsilon,t}) \le 2\varepsilon^s$. Up to a translation, we can assume that $x_{\varepsilon,t} = te_d$ to simplify our notation. We can then define a family of 1-dimensional functions $(f_\varepsilon^{t,\xi})_{\varepsilon>0}$ which we can compare with the optimal 1D profile from Prop. 3.1 such that

$$1 - f_\varepsilon^{t,\xi} \in W_0^{1,p}(\mathbb{R}_+), \text{ and } f_\varepsilon^{t,\xi}(l) = \varphi_\varepsilon\left(te_d + \varepsilon l \xi\right), \text{ for } l \in \left[0, \frac{\bar{l}}{\varepsilon}\right],$$

where $\bar{l}$ is some point close to $\delta' r$ such that $\varphi_\varepsilon(\bar{l}\xi + te_d) = f_{t,\xi,\varepsilon}(\bar{l}) \xrightarrow[\varepsilon\to 0]{} 1$.

As in Step 2, the family $(1 - f_{t,\xi,\varepsilon})_{\varepsilon>0}$ is equibounded in $W^{1,p}(\mathbb{R}_+)$ so that up to a subsequence, it converges weakly to some $f_{t,\xi}$. It also holds that

$$f_{t,\xi}(0) = \lim_{\varepsilon\to 0} \varphi_\varepsilon(x_{\varepsilon,t}) = 0 \text{ and } \lim_{l\to\infty} f_{t,\xi}(l) = \lim_{\varepsilon\to 0} \varphi_\varepsilon(te_d + \delta' r\xi) = 1,$$

so that the limit $f_{t,\xi}$ is admissible for the 1D optimization problem (17), and we have that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \int_0^{\frac{\bar{l}}{\varepsilon}} l^{d-2} \left(\frac{1}{p}|f'_{t,\xi,\varepsilon}|^p + \frac{1}{p'}(1 - f_{t,\xi,\varepsilon})^2\right) \mathrm{d}l \\ \ge \liminf_{\varepsilon\to 0} \int_0^{+\infty} l^{d-2} \left(\frac{1}{p}|f'_{t,\xi}|^p + \frac{1}{p'}(1 - f_{t,\xi})^2\right) \mathrm{d}l \ge \lambda_{p,d}\end{aligned}$$

Let us now gather these ingredients to estimate $\mu(\overline{B_r(x_0)})$. From our previous considerations, it follows that

$$\begin{aligned}\mu(\overline{B_r(x_0)}) &\ge \liminf_{\varepsilon\to 0} \frac{1}{\Lambda_{p,d}} \int_{[-\delta r,\delta r]\setminus(a_\varepsilon,b_\varepsilon)} \int_{D_t} \left(\frac{\varepsilon^{p-(d-1)}}{p}|\nabla\varphi_\varepsilon|^p + \frac{\varepsilon^{-(d-1)}}{p'}(1-\varphi_\varepsilon)^2\right) \mathrm{d}\mathscr{H}^{d-1}\mathrm{d}t \\ &\ge \liminf_{\varepsilon\to 0} \frac{1}{\Lambda_{p,d}} \int_{[-\delta r,\delta r]\setminus(a_\varepsilon,b_\varepsilon)} \int_{\mathbb{S}^{d-2}} \int_0^{\frac{\bar{l}}{\varepsilon}} l^{d-2} \left(\frac{1}{p}|f'_{t,\xi,\varepsilon}|^p + \frac{1}{p'}(1 - f_{t,\xi,\varepsilon})^2\right) \mathrm{d}l\mathrm{d}\mathscr{H}^{d-2}\mathrm{d}t \\ &\ge \frac{1}{\Lambda_{p,d}} \int_{[-\delta r,\delta r]} \int_{\mathbb{S}^{d-2}} \underbrace{\liminf_{\varepsilon\to 0} \left(1_{(a_\varepsilon,b_\varepsilon)^c}(t) \int_0^{\frac{\bar{l}}{\varepsilon}} l^{d-2} \left(\frac{1}{p}|f'_{t,\xi,\varepsilon}|^p + \frac{1}{p'}(1 - f_{t,\xi,\varepsilon})^2\right) \mathrm{d}l\right)}_{\ge \lambda_{p,d}} \mathrm{d}\mathscr{H}^{d-2}\mathrm{d}t \\ &\ge \frac{\sigma_{d-2}\lambda_{p,d}}{\Lambda_{p,d}} \delta 2r = \delta 2r.\end{aligned}$$

Where we have used the fact that $b_\varepsilon - a_\varepsilon < d_H(\Sigma_\varepsilon, \Sigma) \to 0$ and the definition of $\Lambda_{p,d}$.

We conclude that $\theta_1^\star(\mu, x) \ge \delta$, where $\delta \in (0,1)$ is arbitrary. Letting $\delta \to 1$ it follows that $\mu \ge \mathscr{H}^1 \llcorner \Sigma$, as well as (28). □

For the $\Gamma - \limsup$ inequality, we will use precisely the approximating sequence φ_ε proposed in Theorem 3.4 for a given Σ. On the other hand, if ν is a probability measure concentrated in Σ, it is not hard to construct a sequence of absolutely continuous measures approximating it, it suffices to take a mollification with a smooth kernel. With this construction, we have already proven that $\mathcal{AT}_p(\varphi_\varepsilon)$ converges to $\mathscr{H}^1(\Sigma)$, the only work that is left is to check that the terms $\mathcal{C}_\varepsilon(\varphi_\varepsilon)$ and $\varepsilon^{-\ell} \int \varphi_\varepsilon \mathrm{d}\nu_\varepsilon$ converge to 0.

Theorem 3.9 (A recovery sequence)**.** *Suppose that Ω satisfies Hypothesis (H1) and that $s > 1$. Then, for any closed $\Sigma \subset \Omega$ such that $\mathscr{H}^1(\Sigma) < \infty$ and $\nu \in \mathscr{P}(\Sigma)$, there exists a family $(\varphi_\varepsilon, \nu_\varepsilon)_{\varepsilon>0} \subset (1 + W_0^{1,p}(\Omega)) \times \mathscr{P}_{ac}(\Omega)$ such that*

$$\varphi_\varepsilon \xrightarrow[\varepsilon\to 0]{L^2(\Omega)} 1, \quad \nu_\varepsilon \xrightarrow[\varepsilon\to 0]{\star} \nu, \quad \mu_\varepsilon \xrightarrow[\varepsilon\to 0]{\star} \mathscr{H}^1 \llcorner \Sigma, \quad \lim_{\varepsilon\to 0} \mathcal{AT}_p(\varphi_\varepsilon) = \mathscr{H}^1(\Sigma) \tag{39}$$

and

$$\mathcal{C}_\varepsilon(\varphi_\varepsilon) = 0, \text{ for all } \varepsilon > 0 \text{ and } \frac{1}{\varepsilon^\ell}\int_\Omega \varphi_\varepsilon \mathrm{d}\nu_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0. \tag{40}$$

Proof. Following the construction Thm. 3.4, it suffices to consider $\Sigma \subset \operatorname{int}\Omega$, otherwise we exploit the star-shape property of Ω to find a sequence $\Sigma_n \subset \operatorname{int}\Omega$ such that $\mathscr{H}^1 \llcorner \Sigma_n \xrightarrow[n\to\infty]{} \mathscr{H}^1 \llcorner \Sigma$ and perform a diagonal extraction argument with the familes of phase fields approximating Σ_n to obtain the desired result for Σ.

Assuming $\Sigma \subset \operatorname{int}\Omega$, we recall the following notation from the proof of Thm. 3.4: $d_\Sigma(x) \stackrel{\text{def.}}{=} \operatorname{dist}(x,\Sigma)$ so that $\Sigma_r \stackrel{\text{def.}}{=} \{x \in \Omega : d_\Sigma(x) \le r\}$.

Let f_p be the optimal profile from Prop. 3.1. If $1 - f_p$ has compact support, the recovery sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ is then defined as in Theorem 3.4 as

$$\varphi_\varepsilon(x) \stackrel{\text{def.}}{=} f_p\left(\frac{d_\Sigma(x) - b_\varepsilon}{\varepsilon}\right), \tag{41}$$

where b_ε will be chosen shortly. If f_p reaches 1 only asymptotically, we can replace f_p with a suitable $f_{p,\varepsilon}$ that attains 1 in finite time. Either way, we have from Thm. 3.4 $\varphi_\varepsilon \in 1 + W_0^{1,p}(\Omega)$. Since f_p is increasing and continuous, we have that

$$\{\varphi_\varepsilon \le 2\varepsilon^s\} = \left\{d_\Sigma(\cdot) \le b_\varepsilon + \varepsilon {f_p}^{-1}(2\varepsilon^s)\right\} = \{d_\Sigma(\cdot) \le 2\varepsilon^s\} \tag{42}$$

for $b_\varepsilon \stackrel{\text{def.}}{=} 2\varepsilon^s - \varepsilon f_p^{-1}(2\varepsilon^s)$. From the Hölder continuity of f_p, $b_\varepsilon \ge 0$ for ε small enough and $b_\varepsilon = o(\varepsilon)$.

It follows from Thm. 3.4 that $\varphi_\varepsilon \xrightarrow[\varepsilon\to 0]{L^2(\Omega)} 1$, its corresponding family of diffuse transition measures is such that $\mu_\varepsilon \xrightarrow[\varepsilon\to 0]{\star} \mathscr{H}^1 \llcorner \Sigma$ and $\lim_{\varepsilon\to 0} \mathcal{AT}_p(\varphi_\varepsilon, \nu_\varepsilon) = \mathscr{H}^1(\Sigma)$. For the family $(\nu_\varepsilon)_{\varepsilon>0}$ let $(\eta_t)_{t>0}$ be a sequence of mollifiers $\eta_t = t^{-d}\eta\left(\frac{\cdot}{t}\right)$, with η supported at the unitary ball and set $\nu_\varepsilon \stackrel{\text{def.}}{=} \eta_{c_\varepsilon} \star \nu$, for c_ε small enough so that

$$f_p\left(\frac{c_\varepsilon - b_\varepsilon}{\varepsilon}\right) \le \varepsilon^{2\ell} \text{ and } 0 \le c_\varepsilon - b_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0.$$

It then holds that $\nu_\varepsilon \xrightarrow[\varepsilon\to 0]{\star} \nu$, see [4, Thm. 2.2], and $\operatorname{supp}\nu_\varepsilon \subset \Sigma_{c_\varepsilon}$.

To finish the proof, it only remains to show (40). First notice that as ν_ε is concentrated in Σ_{c_ε}, it holds that

$$\frac{1}{\varepsilon^\ell}\int_\Omega \varphi_\varepsilon \mathrm{d}\nu_\varepsilon = \frac{1}{\varepsilon^\ell}\int_\Omega f_p\left(\frac{d_\Sigma(x) - b_\varepsilon}{\varepsilon}\right)\mathrm{d}\nu_\varepsilon \le \frac{1}{\varepsilon^\ell} f_p\left(\frac{c_\varepsilon - b_\varepsilon}{\varepsilon}\right) \le \varepsilon^\ell \xrightarrow[\varepsilon\to 0]{} 0.$$

To compute the term $\mathcal{C}_\varepsilon(\varphi_\varepsilon)$, observe that, from the connectedness of Σ, the set $\{\varphi_\varepsilon \le 2\varepsilon^s\}$ is connected. Given any two points in $\{\varphi_\varepsilon \le 2\varepsilon^s\}$, we project each one onto Σ, since Σ is itself connected, we can find a path in Σ connecting the projections. From (42), the union of these three arcs forms a path inside $\{\varphi_\varepsilon \le 2\varepsilon^s\}$ connecting the two original points. Since inside this level set $F_\varepsilon \circ \varphi_\varepsilon \equiv 0$ by construction, for any two points x, y in this level set, we conclude that $d^{F_\varepsilon\circ\varphi_\varepsilon}(x,y) = 0$. Since the connectedness functional can be written as

$$\mathcal{C}_\varepsilon(\varphi_\varepsilon) = \int_{\{\varphi_\varepsilon \le 2\varepsilon^s\}\times\{\varphi_\varepsilon \le 2\varepsilon^s\}} \beta_\varepsilon(\varphi_\varepsilon(x))\beta_\varepsilon(\varphi_\varepsilon(y)) d^{F_\varepsilon\circ\varphi_\varepsilon}(x,y)\mathrm{d}x\mathrm{d}y,$$

one has $\mathcal{C}_\varepsilon(\varphi_\varepsilon) = 0$ for all $\varepsilon > 0$. □

4. The Γ-convergence: approximation of (ADM) and ($\overline{W\mathscr{H}^1}$)

In this section we finally profit from the general analysis done previously to study the problems we were interested in the first place. We first use the properties of the connectedness functional proved in Lemma 3.6 to show existence of minimizers to the phase-field approximations $\mathcal{AD}_\varepsilon$ and $\overline{\mathcal{WH}^1}_\varepsilon$, defined in the introduction in (5) and (6), and proceed to the proof of our main Theorems 1.1 and 1.2.

Theorem 4.1. *For $\varepsilon > 0$ fixed, both functional $\mathcal{AD}_\varepsilon$ and $\mathcal{WH}^1_\varepsilon$ admit minimizers.*

Proof. First notice that both

$$\inf_{(\nu,\varphi)} \mathcal{AD}_\varepsilon(\nu,\varphi) \text{ and } \inf_{(\alpha,\nu,\varphi)} \mathcal{WH}^1_\varepsilon(\alpha,\nu,\varphi)$$

are finite. This can be seen by considering for instance the recovery sequence from Thm. 3.9 of a segment contained in Ω. Now we can apply the direct method of the calculus of variations to both functionals.

Starting with $\mathcal{AD}_\varepsilon$, let $(\nu_n,\varphi_n)_{n\in\mathbb{N}}$ be a minimizing sequence. Since the infimum of $\mathcal{AD}_\varepsilon$ is finite, it follows that

$$\sup_{n\in\mathbb{N}} \mathcal{AT}_p(\varphi_n) < +\infty,$$

and hence φ_n is bounded in $W^{1,p}(\Omega)$. From Morrey's inequality this sequence is equi-continuous, and it can be taken to be uniformly bounded since the energy can be reduced by thresholding them with the constant 1. From Ascoli-Arzela, it converges uniformly and in $W^{1,p}(\Omega)$, up to a subsequence, to some φ. Similarly, using Banach-Alaoglu we can extract a subsequence such that ν_n converges weakly to some measure ν. We than have that

$$\begin{aligned}
&W_q^p(\rho_0,\nu) = \lim_{n\to\infty} W_q^p(\rho_0,\nu), && \text{from the weak continuity of } W_q\\
&\mathcal{AT}_p(\varphi) \le \liminf_{n\to\infty} \mathcal{AT}_p(\varphi_n), && \text{from the weak convergence in } W^{1,p}(\Omega)\\
&\mathcal{C}_\varepsilon(\nu) = \lim_{n\to\infty} \mathcal{C}_\varepsilon(\nu), && \text{since } \mathcal{C}_\varepsilon \text{ is } C^0 \text{ for uniform convergence, Lemma 3.6}\\
&\int_\Omega \varphi \mathrm{d}\nu = \lim_{n\to\infty} \int_\Omega \varphi_n \mathrm{d}\nu_n, && \text{since } \nu_n \rightharpoonup \text{ in } \mathscr{P}(\Omega) \text{ and } \varphi_n \to \varphi \text{ uniformly.}
\end{aligned}$$

From the fact that (ν_n,φ_n) is a minimizing sequence, it follows that (ν,φ) attains the infimum of $\mathcal{AD}_\varepsilon$.

For $\mathcal{WH}^1_\varepsilon$, a for a minimizing $(\alpha_n,\nu_n,\varphi_n)_{n\in\mathbb{N}}$, a similar argument from the previous case, we can assume up to a subsequence that

$$\begin{aligned}
&\alpha_n \xrightarrow[n\to\infty]{} \alpha, &&\\
&\nu_n \xrightarrow[n\to\infty]{} \nu, && \text{weakly in } L^2(\Omega) \text{ and } \mathscr{P}(\Omega)\\
&\varphi_n \xrightarrow[n\to\infty]{} \varphi, && \text{weakly in } W^{1,p}(\Omega) \text{ and uniformly.}
\end{aligned}$$

And the same continuity and lower semi-continuity property let us conclude that (α,ν,φ) is optimal. □

4.1. **Proof of Γ-convergence for average distance minimizers.** Now we are in position to prove the Γ-convergence result for the average distance minimizers problem as direct consequence of Theorems 3.8 and 3.9.

Proof of Theorem 1.1: Starting with the $\Gamma-\liminf$, let $(\varphi_\varepsilon,\nu_\varepsilon)_{\varepsilon>0}$ such that $\varphi_\varepsilon \xrightarrow[\varepsilon\to0]{L^2(\Omega)} \varphi$ and $\nu_\varepsilon \xrightarrow[\varepsilon\to0]{\star} \nu$. Suppose w.l.o.g. that $\liminf_{\varepsilon\to0} \mathcal{AD}_\varepsilon(\varphi_\varepsilon,\nu_\varepsilon) < \infty$. Up to taking a subsequence attaining the lim inf, it holds that

$$F_\varepsilon(\varphi_\varepsilon,\nu_\varepsilon) \le C \text{ for all } \varepsilon > 0.$$

Once again up to subsequences, it follows from Theorem (3.8) that $\varphi \equiv 1$ and there exists a countably $\mathscr{H}^1$-rectifiable set Σ such that $\operatorname{supp}\nu \subset \Sigma$. In particular, this implies that the Steiner tree connecting $\operatorname{supp}\nu$ exists and has a finite length, [33], since we have that $\mathscr{H}^1_S(\operatorname{supp}\nu) \leq \mathscr{H}^1(\Sigma)$. From the lower semi-continuity of the Wasserstein distance w.r.t. weak convergence and the previous properties it holds that

$$\begin{aligned} W_q^q(\rho_0, \nu) + \Lambda \mathscr{H}^1_S(\operatorname{supp}\nu) \leq W_q^q(\rho_0, \nu) + \Lambda \mathscr{H}^1(\Sigma) &\leq \liminf_{\varepsilon\to 0} \Big(W_q^q(\rho_0, \nu_\varepsilon) + \Lambda \mathcal{AT}_p(\varphi_\varepsilon)\Big) \\ &\leq \liminf_{\varepsilon\to 0} \mathcal{AD}_\varepsilon(\varphi_\varepsilon, \nu_\varepsilon). \end{aligned}$$

For the $\Gamma - \limsup$, for some $\nu \in \mathscr{P}(\Omega)$, suppose that $\mathscr{H}^1_S(\operatorname{supp}\nu) < +\infty$, otherwise there is nothing to prove. This implies that there exists a Steiner tree $\mathcal{S}(\operatorname{supp}\nu)$ attaining the infimum $\mathscr{H}^1_S(\operatorname{supp}\nu)$ with finite length and is hence a countably $\mathscr{H}^1$-rectifiable set. We can then use the recovery sequence proposed in Theorem 3.9 with $\Sigma = \mathcal{S}(\operatorname{supp}\nu)$. As we are in a bounded domain, the Wasserstein distance is continuous for the weak convergence of measures and it holds that

$$\mathcal{AD}_\varepsilon(\varphi_\varepsilon, \nu_\varepsilon) = W_q^q(\rho_0, \nu_\varepsilon) + \Lambda \mathcal{AT}_p(\varphi_\varepsilon) \xrightarrow[\varepsilon\to 0]{} W_q^q(\rho_0, \nu) + \Lambda \mathscr{H}^1_S(\operatorname{supp}\nu).$$

To finish the proof we verify that

$$\min_{\Sigma}\,(\text{ADM}) = \min_{(\nu,\varphi)} \mathcal{AD}.$$

Let Σ and ν be minimizers of (ADM) and $\mathcal{AD}$, respectively, whereas let $\mathcal{S}(\operatorname{supp}\nu)$ and π_Σ denote a Steiner tree of $\operatorname{supp}\nu$ and a measurable selection of the projection operator onto Σ. It holds that

$$\begin{aligned} \min\,(\text{ADM}) &\leq \int_\Omega \operatorname{dist}(x, \mathcal{S}(\operatorname{supp}\nu))^q \mathrm{d}\rho_0(x) + \Lambda \mathscr{H}^1_S(\operatorname{supp}\nu) \\ &= \int_\Omega \operatorname{dist}(x, \operatorname{supp}\nu)^q \mathrm{d}\rho_0(x) + \Lambda \mathscr{H}^1_S(\operatorname{supp}\nu) \\ &\leq W_q^q(\rho_0, \nu) + \Lambda \mathscr{H}^1_S(\operatorname{supp}\nu) = \min \mathcal{AD} \\ &\leq W_q^q(\rho_0, (\pi_\Sigma)_\sharp \rho_0) + \Lambda \mathscr{H}^1(\Sigma) \\ &= \int_\Omega \operatorname{dist}(x, \Sigma)^q \mathrm{d}\rho_0(x) + \Lambda \mathscr{H}^1(\Sigma) = \min\,(\text{ADM}). \end{aligned}$$

It is then clear that if there is ν optimal for $\mathcal{AD}$ then $\mathcal{S}(\operatorname{supp}\nu)$ is optimal for (ADM) and similarly, if is Σ optimal for (ADM) then $(\pi_\Sigma)_\sharp \rho_0$ is optimal for $\mathcal{AD}$. Let us prove the converse of these propositions.

If ν is optimal and cannot be written this way, then $\Sigma = \mathcal{S}(\operatorname{supp}\nu)$ is a minimizer and $W_q^q(\rho_0, \nu) > \int_\Omega \operatorname{dist}(x, \Sigma)^q \mathrm{d}\rho_0$, otherwise it would follow necessarily that $\nu = (\pi_\Sigma)_\sharp \rho_0$, hence

$$\mathcal{AD}(\nu, \varphi \equiv 1) > \int_\Omega \operatorname{dist}(x, \Sigma)^q \mathrm{d}\rho_0 + \Lambda \mathscr{H}^1(\Sigma) \geq \min \mathcal{AD}$$

which contradicts the minimality of ν.

Similarly suppose that Σ is optimal and cannot be written as the Steiner tree of the support of any minimizer ν. We know that $\nu' = (\pi_\Sigma)_\sharp \rho_0$ is a minimizer whose support is contained in Σ, so

$$\min\,(\text{ADM}) = W_q^q(\rho_0, \nu') + \Lambda \mathscr{H}^1(\Sigma) > W_q^q(\rho_0, \nu') + \Lambda \mathscr{H}^1_S(\operatorname{supp}\nu') \geq \min \mathcal{AD},$$

contradicting the minimality of Σ. ■

4.2. **Proof of Γ-convergence for $(\overline{W\mathscr{H}^1})$.** Now we move on to the Γ-convergence result for the problem $(\overline{W\mathscr{H}^1})$. We shall use two results from the theory developed for this problem in [15]. Recall that the relaxed problem is stated in terms of the length functional defined as

$$\mathcal{L}(\nu) \stackrel{\text{def.}}{=} \inf\left\{\alpha \geq 0 : \alpha\nu \geq \mathscr{H}^1 \llcorner \operatorname{supp}\nu\right\}, \tag{43}$$

which is the l.s.c. relaxation of the functional $\nu_\Sigma \mapsto \mathscr{H}^1(\Sigma)$ if ν_Σ is the probability measure uniformly distributed over a connected set Σ, *i.e.* $\nu_\Sigma = \frac{1}{\mathscr{H}^1(\Sigma)}\mathscr{H}^1 \llcorner \Sigma$ and $+\infty$ otherwise. The first lemma we shall introduce is an approximation result for measures ν such that $\mathcal{L}(\nu) < +\infty$, which is the crucial result in order to understand $(\overline{W\mathscr{H}^1})$ as the relaxation of $(W\mathscr{H}^1)$.

Lemma 4.2 ([15])**.** *Let $\nu \in \mathscr{P}(\Omega)$ such that $\mathcal{L}(\nu) < \infty$, there exists a sequence of connected sets $(\Sigma_n)_{n\in\mathbb{N}}$ such that*

- $\Sigma_n \xrightarrow[n\to\infty]{d_H} \Sigma$ *and* $\mathscr{H}^1(\Sigma_n) \xrightarrow[n\to\infty]{} \mathcal{L}(\nu)$*;*
- $\nu_{\Sigma_n} \stackrel{\text{def.}}{=} \dfrac{1}{\mathscr{H}^1(\Sigma_n)}\mathscr{H}^1 \llcorner \Sigma_n \xrightharpoonup[n\to\infty]{} \nu.$

As we have discussed, our Γ-convergence result actually approximates the relaxed problem $(\overline{W\mathscr{H}^1})$ instead of $(W\mathscr{H}^1)$, but we cannot expect anything better since the corresponding energy of the original problem is not l.s.c. and Γ-limits are always l.s.c. [6]. However, under certain conditions, the minimizers of the relaxed problem are known to be minimizers of the original problem, this is the content of the following result.

Lemma 4.3 ([15])**.** *Let $\Omega \subset \mathbb{R}^d$ be a compact and convex domain with non-empty interior, for any $\rho_0 \in \mathscr{P}(\Omega)$ that does not charge $\mathscr{H}^1$-rectifiable sets, any minimizer of $(\overline{W\mathscr{H}^1})$ is uniformly distributed, and hence its support solves $(W\mathscr{H}^1)$.*

Proof of Theorem 1.2: Let us start with the $\Gamma-\liminf$ inequality. Consider $(\alpha_\varepsilon, \nu_\varepsilon, \varphi_\varepsilon) \xrightarrow[\varepsilon\to 0]{} (\alpha, \nu, \varphi)$ in the product topology of $\mathbb{R}$, weak convergence of measures and strong $L^2(\Omega)$ convergence. Up to extracting a subsequence in ε, we can suppose w.l.o.g. that there is a constant $C > 0$ such that $\mathcal{WH}^1_\varepsilon(\alpha_\varepsilon, \nu_\varepsilon, \varphi_\varepsilon) \leq C$, otherwise there is nothing to prove. This can be done by first assuming the lim inf is finite and taking a subsequence attaining it. Clearly from the continuity of the Wasserstein distance, it holds that

$$W_q^q(\rho_0, \nu_\varepsilon) + \Lambda\alpha_\varepsilon \xrightarrow[\varepsilon\to 0]{} W_q^q(\rho_0, \nu) + \Lambda\alpha,$$

hence to conclude, we must verify the constraints $\varphi \equiv 1$ and $\alpha\nu \geq \mathscr{H}^1 \llcorner \operatorname{supp}\nu$, where $\operatorname{supp}\nu$ is connected.

Recalling that μ_ε is the diffuse transition measure defined at (7) with the function φ_ε, notice that

$$\|\alpha_\varepsilon\nu_\varepsilon - \mu_\varepsilon\|_{\mathcal{M}(\Omega)} = \|\alpha_\varepsilon\nu_\varepsilon - \mu_\varepsilon\|_{L^1(\Omega)} \leq |\Omega|^{1/2}\, \|\alpha_\varepsilon\nu_\varepsilon - \mu_\varepsilon\|_{L^2(\Omega)} \leq (C|\Omega|\varepsilon)^{1/2}.$$

Therefore, $\alpha_\varepsilon\nu_\varepsilon - \mu_\varepsilon$ converge to 0 strongly, and hence $\mu_\varepsilon \xrightharpoonup[\varepsilon\to 0]{\star} \alpha\nu$. It also holds that

$$\mathcal{AT}_p(\varphi_\varepsilon) = \mu_\varepsilon(\Omega) \leq \|\alpha_\varepsilon\nu_\varepsilon - \mu_\varepsilon\|_{L^1(\Omega)} + \|\alpha_\varepsilon\nu_\varepsilon\|_{L^1(\Omega)} \leq \alpha_\varepsilon + (C|\Omega|\varepsilon)^{1/2},$$

so we can find another constant $C' > 0$ such that

$$F_\varepsilon(\varphi_\varepsilon, \nu_\varepsilon) = \mathcal{AT}_p(\varphi_\varepsilon) + \frac{1}{\varepsilon^\kappa}\mathcal{C}_\varepsilon(\varphi_\varepsilon) + \frac{1}{\varepsilon^\ell}\int_\Omega \varphi_\varepsilon \mathrm{d}\nu_\varepsilon \leq C' \text{ for all } \varepsilon > 0.$$

From Theorem (3.8), the sequence φ_ε converges strongly in $L^2(\Omega)$ to the constant 1, and there is a connected, countably $\mathscr{H}^1$-rectifiable set Σ such that $\operatorname{supp}\nu \subset \Sigma \subset \operatorname{supp}\mu$

and $\mu = \alpha\nu \geq \mathscr{H}^1 \llcorner \Sigma$. Hence, since $\mu = \alpha\nu$ and $\alpha\nu \geq \mathscr{H}^1 \llcorner \operatorname{supp}\nu$ we have that $\operatorname{supp}\nu = \Sigma = \operatorname{supp}\mu$ and $\alpha \geq \mathcal{L}(\nu)$, implying

$$W_q^q(\rho_0,\nu) + \Lambda\mathcal{L}(\nu) \leq W_q^q(\rho_0,\nu) + \Lambda\alpha \leq \liminf_{\varepsilon\to 0}\Big(W_q^q(\rho_0,\nu_\varepsilon) + \Lambda\alpha_\varepsilon\Big)$$
$$\leq \liminf_{\varepsilon\to 0} \mathcal{WH}^1(\alpha_\varepsilon,\nu_\varepsilon,\varphi_\varepsilon).$$

Moving on to the construction of the recovery sequence, our strategy is to combine Lemma 4.2 with the fact that the diffuse transition measures μ_ε related to the recovery sequence from Theorem (3.9) converge to uniform measures of the form $\mathscr{H}^1 \llcorner \Sigma$.

Given $\alpha\nu \geq \mathscr{H}^1 \llcorner \operatorname{supp}\nu$ such that ν is a probability measure, $\operatorname{supp}\nu$ is connected and $0 < \alpha = \mathcal{L}(\nu) < +\infty$, since if $\alpha = \mathcal{L}(\nu) = 0$ then ν is concentrated in a single point. For clarity of notation set $\Sigma = \operatorname{supp}\nu$ and let Σ_n be the approximating sequence from Lemma (4.2). For each $n \in \mathbb{N}$, construct the recovery sequence $(\varphi_{n,\varepsilon})_{\varepsilon>0}$ from Theorem (3.9), built from the set Σ_n. From the construction, $\mathcal{C}_\varepsilon(\varphi_{n,\varepsilon}) = 0$ and it holds that

$$\mu_{n,\varepsilon} \xrightarrow[\varepsilon\to 0]{\star} \mathscr{H}^1 \llcorner \Sigma_n, \text{ for all } n \in \mathbb{N},$$

where $\mu_{n,\varepsilon}$ is the diffuse transition measure associated with $\varphi_{n,\varepsilon}$. Define

$$\alpha_{n,\varepsilon} \overset{\text{def.}}{=} \mu_{n,\varepsilon}(\Omega) \text{ and } \nu_{n,\varepsilon} \overset{\text{def.}}{=} \frac{1}{\alpha_{n,\varepsilon}}\mu_{n,\varepsilon} \in \mathscr{P}(\Omega),$$

so that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\alpha_{n,\varepsilon} = \lim_{n\to\infty}\mathscr{H}^1(\Sigma_n) = \alpha,$$
$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\nu_{n,\varepsilon} = \lim_{n\to\infty}\frac{1}{\mathscr{H}^1(\Sigma_n)}\mathscr{H}^1 \llcorner \Sigma_n = \nu.$$

With a diagonal argument, we select a decreasing sequence $\varepsilon_n \to 0$ such that

$$\mathcal{C}_{\varepsilon_n}(\varphi_{n,\varepsilon_n}) = \|\alpha_{n,\varepsilon_n}\nu_{n,\varepsilon_n} - \mu_{n,\varepsilon_n}\|_{L^2(\Omega)} = 0 \text{ and } \alpha_{n,\varepsilon_n}, \nu_{n,\varepsilon_n} \to \alpha, \nu.$$

Our recovery sequence is then defined as

$$(\alpha_\varepsilon,\nu_\varepsilon,\varphi_\varepsilon) \overset{\text{def.}}{=} (\alpha_{n,\varepsilon_n},\nu_{n,\varepsilon_n},\varphi_{n,\varepsilon_n}) \text{ if } \varepsilon_n \leq \varepsilon < \varepsilon_{n-1},$$

so the continuity of the Wasserstein distance yields

$$\mathcal{WH}^1_\varepsilon(\alpha_\varepsilon,\nu_\varepsilon,\varphi_\varepsilon) \xrightarrow[\varepsilon\to 0]{} W_q^q(\rho_0,\nu) + \Lambda\mathcal{L}(\nu),$$

and the result follows.

The fact that, whenever ρ_0 does not charge 1-dimensional sets, cluster points of minimizers of $\mathcal{WH}^1_\varepsilon$ converge to a measure ν_Σ, where Σ minimizes the original problem $(W\mathscr{H}^1)$ follows from the fundamental property of Γ-convergence and the fact that under these conditions, from Lemma 4.3, minimizers of the relaxed problem are induced from minimizers of the original. ■

5. Conclusion

In this paper we have discussed a general approach to define phase-field approximations for the Wasserstein-$\mathscr{H}^1$ problem as well as the average distance minimizers problem, the key ingredients being the interplay between the Ambrosio-Tortorelli functional $\mathcal{AT}_p$ and the connectivity functional $\mathcal{C}_\varepsilon$ in the general Thm. 3.8 and the approximation properties of the diffuse transition measures in the weak topology from Thm. 3.4. These results can be easily applied to other approximation schemes. One example that would not be as simple is the general Monge-Kantorovitch model proposed in [14] since the network Σ appears in the definition of a metric that is used inside an optimization problem, a 1-Wasserstein distance.

Many questions are left unanswered, on the theoretical side, recalling that the original model of Modica and Mortola was motivated by the Cahn-Hillard equations, one could ask if there is a connection between a modified model with a p-Laplacian and a suitable family of p-elliptic functionals as $\mathcal{AT}_p$ employed in the present work. Also inspired on previous phase-field models, one could ask if optimal or almost optimal phase-fields enjoy some sort of equipartition of energy. We forced this to be the case in the recovery sequence constructed in Thm. 3.4, but it might be a more general phenomenon.

Numerical implementations of the approximations will be investigated in future work and might serve as a source of conjectures for theoretical questions and qualitative properties about both the Wasserstein-$\mathscr{H}^1$ and the average distance minimization problems.

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