ABSENSE OF LOOPS FOR THE WASSERSTEIN-$\mathscr{H}^1$ PROBLEM: THE LOCALIZATION/BLOW-UP ARGUMENT

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Abstract. In the present work we prove that minimizers of the Wasserstein-$\mathscr{H}^1$ problem, introduced recently in [5], are trees in two cases: when the target measure is a sum of finitely many Dirac masses or when it has a bounded density.

Contents

1. Introduction

Consider the following problem: given a probability measure $\varrho_0 \in \mathscr{P}(\mathbb{R}^d)$, how can it best be approximated with a 1-dimensional set, that is how can we approximate it with a measure uniformly distributed over such lower dimensional sets? This question has been recently addressed with a variational approach in [5] with the following variational problem:

$$(P_\Lambda) \qquad \inf_{\Sigma \text{ connected}} W_p^p\left(\varrho_0, \frac{1}{\mathscr{H}^1(\Sigma)}\mathscr{H}^1 \llcorner \Sigma\right) + \Lambda \mathscr{H}^1(\Sigma),$$

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where W_p corresponds to the Wasserstein distance, defined via the value of an optimal transport problem [1, 15, 17], that metrizes the weak convergence of probability measures and $\mathscr{H}^1$ denotes the 1-dimensional Hausdorff measure [2, 9]. Notice that the penalization of the total length is necessary otherwise the Wasserstein distance could be made arbitrarily small by choosing a suitable space-filling curve, whereas without the connectedness constraint the same could be achieved by approximating ρ_0 with a sequence of atomic measures, while have zero length.

In [5] existence of an optimal network Σ has been proven, provided that the regularization parameter Λ is small enough and that ρ_0 does not give mass to 1-dimensional sets. Afterwards the qualitative properties of this problem have been studied, still in [5] minimizers are shown to be Ahlfors regular; while in [8] a phase-field approximation result for (P_Λ) has been derived with an Ambrosio-Tortorelli type functional. The goal of this work is to show that optimal networks are trees, *i.e.* none of its subsets is homeomorphic to $\mathbb{S}^1$.

Differently from other similar problems, such the Steiner [3, 13], or the average distance minimizers problem [4, 12], existence of an optimal network to (P_Λ) does not follows directly from the *Direct Method of the Calculus of Variations*. The difficulty stems from the lack of compatibility between the convergence of sets (Hausdorff convergence) and the narrow convergence of measures, see Section 2 for more details on such notions of convergence. Indeed, cluster points for sequences of the form $\mathscr{H}^1 \llcorner \Sigma_n$ are not necessarily of the form $\mathscr{H}^1 \llcorner \Sigma$ due to concentration of mass effects.

For this reason, its lower semi-continuous relaxation is introduced, for which existence of minimizers can be easily shown with the direct method. It can be written as

$$(\overline{P}_\Lambda) \qquad \inf_{\nu \in \mathscr{P}_p(\mathbb{R}^d)} W_p^p(\rho_0, \nu) + \Lambda \mathcal{L}(\nu)$$

where the *length functional* $\mathcal{L}$ is defined for a probability measure $\nu \in \mathscr{P}(\mathbb{R}^d)$ as

$$\mathcal{L}(\nu) \overset{\text{def.}}{=} \min\left\{ \alpha \geq 0 : \begin{array}{c} \alpha\nu \geq \mathscr{H}^1 \llcorner \operatorname{supp} \nu \\ \text{if } \operatorname{supp} \nu \text{ is connected.} \end{array} \right\}, \tag{1.1}$$

which is the l.s.c. relaxation of the functional defined by $\dfrac{1}{\mathscr{H}^1(\Sigma)} \mathscr{H}^1 \llcorner \Sigma \mapsto \mathscr{H}^1(\Sigma)$, if Σ is connected, and $+\infty$ otherwise. For more details and properties on the length functional, the reader is referred to [5] where it was first introduced, or to Section 2.3 for a brief discussion.

With this new formulation of the problem, the proof of existence consists off showing that any minimizer of $(\overline{P}_\Lambda)$ is uniformly distributed over this support, being therefore a solution to (P_Λ). Heuristically this can be easily done; suppose that ν is a minimizer of $(\overline{P}_\Lambda)$, if it has an excess, that is regions where its density is not constant, it can be proved that this excess measure is formed through projections onto Σ. Therefore, in principle one could construct a better competitor with a constant density by replacing any excess of the uniform density with segments in the opposite direction of the projections, as represented in Figure 1.

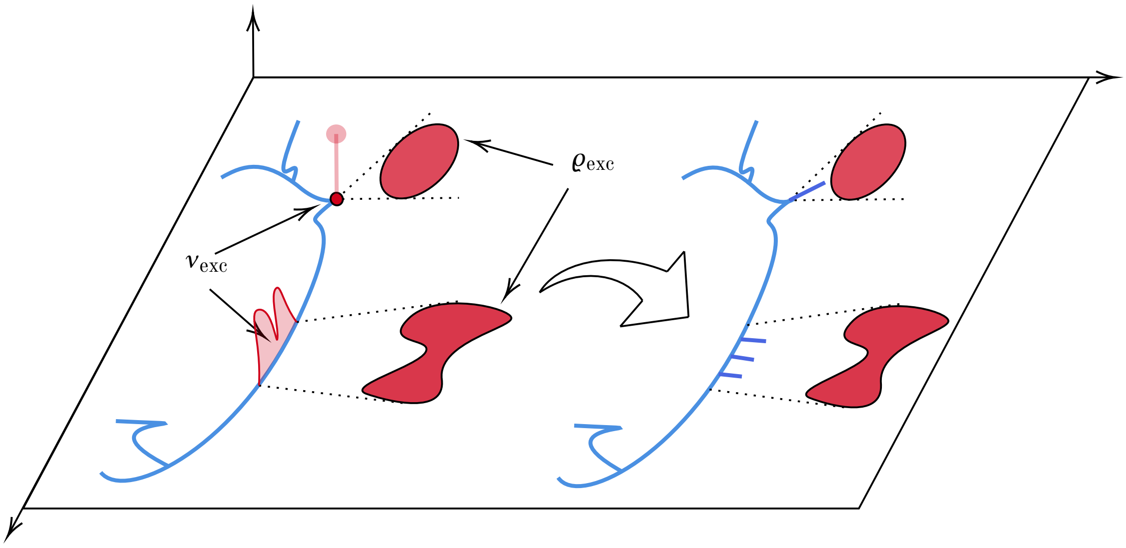

FIGURE 1. Heuristic proof of existence of an optimal shape for problem (P_Λ). If a solution has an excess part, represented in the figure by a measure having a density along Σ and a Dirac mass, it must be formed through projections onto Σ. But then it is better to send the excess mass that is being projected to small segments in the direction of the projection.

However, since we lack much information on the measure that is projected to form the excess, it is unclear *a priori* how to select to which directions should point the segments that decrease the energy. For this reason, in [5] a *localization/blow-up* argument is developed, that yields a localized problem which inherits the projection property. In the blow-up limit, the optimal network Σ is replaced by its *approximate tangent space* $T_{y_0}\Sigma$ (see Section 2.3) at a carefully chosen point y_0. This simplifies the construction of a better competitor since now all projection directions are orthogonal to $T_{y_0}\Sigma$.

In principle, the localization/blow-up argument can be carried out for any structure that is formed via projections onto the optimal network. As a result, if we can prove that loops are formed through projections, one could also expect that optimal networks should not have them, with a similar heuristic from the question of existence. Indeed, we show that if a loop exists, it must be formed via projections, hence one can localize around a carefully chosen point and "open" the loop, while adding a structure that reduces the cost of projecting onto Σ, see Figure 2. Once again, conducting this argument directly is not simple since we cannot control the direction of projection onto the loop, therefore we implement a variation of the the *localization/blow-up argument* that is described in more detail in the sequel.

1.1. **Contributions and the localization/blow-up argument.** As previously stated, in this work we show that the support of minimizers of $(\overline{P}_\Lambda)$ are trees in two cases

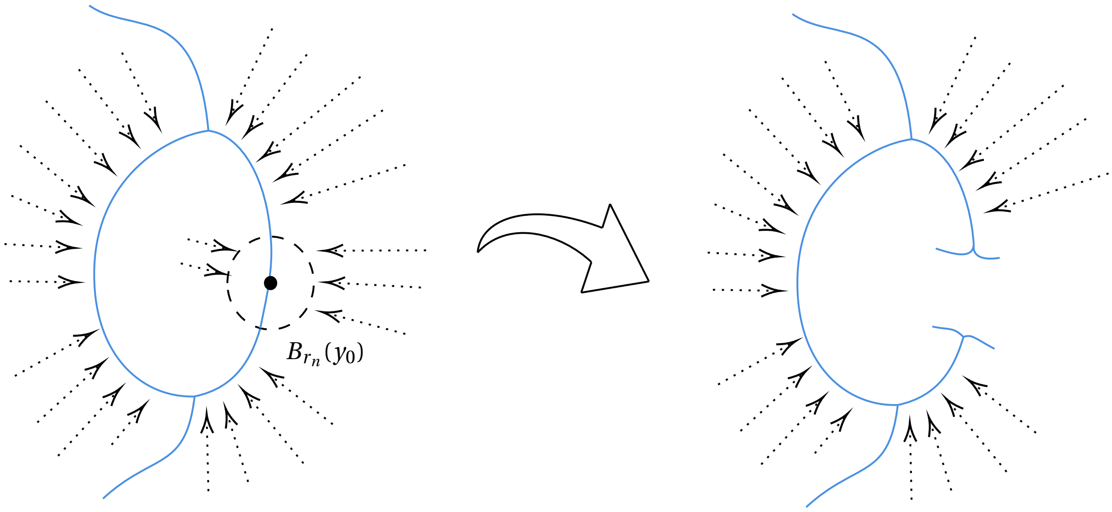

Figure 2. Argument for absence of loops for $(\overline{P}_\Lambda)$. As in the proof of existence, we begin by showing that loops are formed through projections and later use this information to construct a better competitor.

Case 1: if ϱ_0 is a convex combination of Dirac masses, *i.e.*

$$\varrho_0 = \mu_N = \sum_{i=1}^N a_i \delta_{x_i}, \text{ for } \sum_{i=1}^N a_i = 1.$$

Case 2: ϱ_0 is absolutely continuous w.r.t. the Lebesgue measure with compact support and bounded density, *i.e.* $\varrho_0 \in L^\infty(\mathbb{R}^d)$.

Under these hypotheses, we can apply the localization/blow-up argument, also used in [5] for the existence of optimal networks to (P_Λ). More generally, it could be used to rule out the appearance of any structure that is formed through projections. Hopefully this strategy of proof can prove to be useful in other contexts, so in the sequel we go through each step.

(1) **Identify a structure that is formed through projections:** In the first step one proves that the structure one wishes to exclude is formed via projections of the initial measure ϱ_0 using the optimal transport problem in the energy from $(\overline{P}_\Lambda)$. Such structures can be loops or the excess measure, mentioned above for the proof of existence.

(2) **Chose a point y_0 with good properties to localize:** The next step is to select a point y_0 from this structure (inside the loop, or on the support of the excess measure) for which we can make variations, for instance such that the approximate tangent space $T_{y_0}\Sigma$ exists, and that is a non-cut point for the absence of loops, allowing to remove a neighborhood of it without breaking the connectedness.

(3) **Define localized problems and show they Γ-converge:** In the sequel, we must be able to craft variations that are localized around $\Sigma \cap B_{r_n}(y_0)$ which remain admissible. These variations define a family of functionals $(F_n)_{n\in\mathbb{N}}$, which is minimized by a localization of the solution to the

original problem. In the sequel, we compute the limit functional F of the sequence F_n in the sense of Γ-convergence.

In the proof of existence, it is necessary that the variations respect the density penalization introduced by the length functional (1.1). In the case of the absence of loops, we must be careful with the connectedness constraint, hence the ball $B_{r_n}(y_0)$ should be chosen so that $\Sigma \setminus B_{r_n}(y_0)$ remains connected.

(4) **Show that the projection property passes to the limit:** In this step, we use the fundamental property of Γ convergence, so that the sequence of localizations that minimize the functionals F_n converge to a minimizer of the limit F. In addition, we also verify that the projection property proved in step (1) is also passed to the limit, so that this minimizer of F is also formed via projections, but this time onto the approximate tangent space $T_{y_0}\Sigma$.

(5) **Construct a better competitor for F:** Finally, we exploit the projection property of the limit to construct a strictly better competitor for the minimization of F. This contradicts the entire construction, and in particular contradicts the existence of the structure from step (1).

This argument is reminiscent of an approach from Santambrogio and Tilli in [16] used to fully characterize the blow-ups of any point from optimal networks for the *average distance functional*, see [7]. In their work, a crucial ingredient was the full topological characterization of such optimal networks done in $\mathbb{R}^2$ since the introduction of the problem by Butazzo and Stepanov in [4], where it was proven that optimizers are trees with finitely many branching points, each one being triple junctions of 120 degrees.

This result has recently been generalized to $\mathbb{R}^d$ in [11]. Their approach consists of defining a vector field, the *barycenter field*, which measures from which direction the mass is on average being projected onto the network. This allows them to develop a local improvement theory of the average distance problem. Adapting these techniques to the Wasserstein-$\mathscr{H}^1$ problem might be an interesting direction of investigation, which can hopefully shed some light onto other topological properties of minimizers for our problem.

1.2. **Structure of this manuscript.** In Section 2 we make a brief review of the basic facts of optimal transport and geometric measure theory, which shall be useful for our analysis. A particular emphasis is given to Section 2.4, where we study a slight refinement of a classical lemma used to prove absence of loops in problems such as the Steiner or the average distance problems.

This refinement might not be surprising to seasoned experts on the field, but is particularly relevant to the implementation of the localization/blow-up argument, which is done in Section 3 and culminates at Theorem 3.5 where we obtain the desired absence of loops. Some proofs therein are postponed to Appendix A, since they are only minor variations of the proofs from [5].

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2. Preliminaries

In this section we recall the notions of convergence of sets and measures required in this article as well as the tools from geometric measure theory that will be employed. Most of the results presented here are well known and are recalled for the sake of readability, as well as to establish notation. Therefore, more experienced readers may want to skip this, expect maybe for Lemma 2.5 from subsection 2.4, which is a small refinement of a result frequently used in the literature to prove absence of loops in 1-dimensional shape optimization problems, see for instance [4, Lemma 6.1]. The usual result says that around every non-cut point one can remove a connected set with diameter as small as we want and still keep the connectedness of the network. This improvement says that such sets can be taken to be the intersection of the network and balls of arbitrarily small radius around the non-cut point, which is very convenient to perform the localization/blow-up argument in the sequel.

2.1. **Convergence of sets and measures.** To formulate variational problems on the space of continua, it is essential to equip this space with a topology that preserves connectedness and finite length. For this, *Hausdorff* and *Kuratowski* convergences are introduced, as detailed in [14]. These convergences are shown to maintain the desired properties when restricted to connected sets with bounded length.

Definition 2.1. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed sets of $\mathbb{R}^d$. If $A \subset \mathbb{R}^d$ is closed, we say that

- A_n converges in the Hausdorff sense to A if $d_H(A_n, A) \xrightarrow[n\to\infty]{} 0$, where d_H is called the Hausdorff distance and is defined as

$$d_H(A,B) \stackrel{\text{def.}}{=} \max\left\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\right\}, \text{ we write } A_n \xrightarrow[n\to\infty]{d_H} A. \tag{2.1}$$

- A sequence of closed sets C_n converges in the sense of Kuratowski to C, and we write $C_n \xrightarrow[n\to\infty]{K} C$, when
  (1) for all sequences $x_n \in C_n$, all its cluster points are contained in C.
  (2) For all points $x \in C$ there exists a sequence $x_n \in C_n$, converging to x.

Furthermore, $A_n \xrightarrow[n\to\infty]{d_H} A$ if and only if $\operatorname{dist}(\cdot, A_n) \xrightarrow[n\to\infty]{} \operatorname{dist}(\cdot, A)$ uniformly. Similarly, Kuratowski convergence corresponds to the agreement of inner and outer limits:

$$\liminf_{n\to\infty} C_n \stackrel{\text{def.}}{=} \left\{x \in \mathbb{R}^d : \limsup_{n\to\infty}\operatorname{dist}(x, C_n) = 0\right\},$$
$$\limsup_{n\to\infty} C_n \stackrel{\text{def.}}{=} \left\{x \in \mathbb{R}^d : \liminf_{n\to\infty}\operatorname{dist}(x, C_n) = 0\right\},$$

in other words Kuratowski convergence holds if and only if $\operatorname{dist}(\cdot, C_n) \to \operatorname{dist}(\cdot, C)$ pointwise. Since the distance functions are 1-Lipschitz, by Ascoli-Arzelà's Theorem we have that

$$C_n \xrightarrow[n\to\infty]{K} C \text{ if and only if } \operatorname{dist}(\cdot, C_n) \xrightarrow[n\to\infty]{} \operatorname{dist}(\cdot, C) \text{ locally uniformly.}$$

As a result, Hausdorff convergence implies Kuratowski convergence, and both notions coincide on compact sets. Importantly, *Blaschke's Theorem*, see [2, Thm. 6.1], states that the Hausdorff topology inherits compactness from the compactness of uniform convergence of the distance functions.

2.2. **Narrow convergence of probability measures and the Wasserstein distances.** Due to Riesz' representation theorem the set of Radon measures $\mathscr{M}(\mathbb{R}^d)$ is known to be the topological dual of the continuous functions. As a result, it is frequently endowed with the *local weak-$\star$* convergence: a sequence $(\mu_n)_{n\in\mathbb{N}}$ is said to converge narrowly to μ in $\mathscr{M}_{loc}(\mathbb{R}^d)$ [2, Def. 1.58] if

$$\int_{\mathbb{R}^d} \phi \mathrm{d}\mu_n \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^d} \phi \mathrm{d}\mu \text{ for all } \phi \in \mathscr{C}_c(\mathbb{R}^d).$$

This notion of convergence however does not preserve the total mass of the sequence $(\mu_n)_{n\in\mathbb{N}}$, as a portion of the mass can be lost at infinity. This is one of the difficulties in implementing Step (4) of the localization/blow-up argument, see the discussion before Lemma 3.2.

For this reason, when working with Radon probability measures it is customary to work with the *narrow topology*, defined by replacing the space of continuous functions with compact support $\mathscr{C}_c(\mathbb{R}^d)$ by the class of continuous and bounded functions $\mathscr{C}_b(\mathbb{R}^d)$. Naturally, if the supports of a convergent sequence $(\mu_n)_{n\in\mathbb{N}}$ are all contained in the same compact subset of $\mathbb{R}^d$, then both notions of convergence coincide and the mass is preserved even under the weak-$\star$ convergence. This will be the case most times in this work, unless when we deal with blow-ups of sets and measures, when it is inevitable to send the support of the measures to infinity.

Nonetheless, the narrow topology is actually metrizable and a possible choice of distance for this topology are the so called *p-Wasserstein distances*[1] defined via the value of an optimal transportation problem(see [1, 15, 17] for more details) as follows: given $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ with finite p-moments, $p \geq 1$, the p-Wasserstein distance is defined as

$$W_p^p(\mu,\nu) \overset{\text{def.}}{=} \min_{\gamma\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \mathrm{d}\gamma(x,y),$$

where $\Pi(\mu,\nu) \overset{\text{def.}}{=} \left\{\gamma \in \mathscr{P}(\mathbb{R}^d\times\mathbb{R}^d) : (\pi_0)_\sharp\gamma = \mu,\ (\pi_1)_\sharp\gamma = \nu\right\}$ corresponds to the couplings with marginals μ and ν. This corresponds to Kantorovitch's formulation of the problem, which is known under certain conditions to actually be a solution to Monge's problem

$$\inf_{T_\sharp\mu=\nu} \int_{\mathbb{R}^d} |x-T(x)|^p \mathrm{d}\mu(x),$$

where the *pushforward measure* is $T_\sharp\mu(A) \overset{\text{def.}}{=} \mu(T^{-1}(A))$, for any Borel set $A \subset \mathbb{R}^d$. The connection between both formulations is give by Brenier's Theorem which states that whenever μ does not give mass to $(d-1)$-dimensional sets,

[1] To be more precise, convergence with respect to the p-Wasserstein distance is equivalent to narrow convergence plus convergence of the p-moments, but the second condition is trivial in compact domains, which will be always the case where this is exploited in this paper.

there is a unique optimal transportation plan that is actually induced by a map, it can be written as $\gamma = (\mathrm{id}, T)_\sharp \mu$.

2.3. **Gołąb's Theorem, the length functional, blow-ups and approximate tangent spaces.** In the sequel, we consider a sequence of continua $(\Sigma_n)_{n\in\mathbb{N}}$ converging to Σ in the sense of Kuratowski. We are mostly interested in the sequence of measures $\mathscr{H}^1 \llcorner \Sigma_n$, up to subsequences, we can always assume it to converge weakly to a measure μ. The classical version of Gołąb's Theorem says that $\mu \geq \mathscr{H}^1 \llcorner \Sigma$, while in [5], this result is proved under the weaker Kuratowski convergence and the sequence Σ_n doesn't have to be bounded, in fact it can have infinite length, as long as it is locally finite.

Theorem 2.2 (Density version of Gołąb's Theorem). *Let* $(\Sigma_n)_{n\in\mathbb{N}}$ *be a sequence of closed and connected subsets of* $\mathbb{R}^d$ *converging in the sense of Kuratowski to some closed set* Σ *and having locally uniform finite length,* i.e. *for all* $R > 0$

$$\sup_{n\in\mathbb{N}} \mathscr{H}^1(\Sigma_n \cap B_R(x_0)) < +\infty.$$

Define the measures $\mu_n \overset{\text{def.}}{=} \mathscr{H}^1 \llcorner \Sigma_n$*, and let* μ *be a weak-*$\star$ *cluster point of this sequence. Then* $\operatorname{supp}\mu \subset \Sigma$ *and it holds that*

$$\mu \geq \mathscr{H}^1 \llcorner \Sigma,$$

in the sense of measures.

This result is central to understand the *length functional* described in the introduction. Consider the functional defined over the space of probability measures as

$$\ell(\nu) \overset{\text{def.}}{=} \begin{cases} \mathscr{H}^1(\Sigma), & \text{if } \nu = \dfrac{1}{\mathscr{H}^1(\Sigma)} \mathscr{H}^1 \llcorner \Sigma, \text{ for } \Sigma \text{ connected}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.2}$$

Using Gołąb's Theorem, one can show that the lower semi-continuous relaxation of the above functional is given by the length functional

$$\mathcal{L}(\nu) \overset{\text{def.}}{=} \begin{cases} \min\left\{\alpha \geq 0 : \alpha\nu \geq \mathscr{H}^1 \llcorner \operatorname{supp}\nu\right\}, & \text{if } \operatorname{supp}\nu \text{ is connected}, \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.3}$$

which is used in the definition of the relaxed formulation ($\overline{P}_\Lambda$) and allows for much more flexibility once creating competitors to optimizers and extract information from them, as for instance in the proof of Proposition 3.1. The challenge associated with this functional is that, as opposed with $\Sigma \mapsto \mathscr{H}^1(\Sigma)$ it has a non-local flavor. Indeed, if we want to *reduce the value* of $\mathcal{L}(\nu)$ we must *increase the* $\mathscr{H}^1$ *density* of ν along all of its support Σ, even if we just want to study the behavior of a small neighborhood of in Σ. This is particularly inconvenient when combined with an optimal transportation cost. On the other hand, adding any structure to Σ, with a smaller density will increase the value of $\mathcal{L}$.

Gołąb's Theorem is also useful to extract a finer information on the blow-ups of 1-rectifiable connected sets. Due to a result from Besicovitch, we know that the connected sets Σ with finite length that are of interest to us are actually countably $\mathscr{H}^1$-rectifiable [2, 9]. In other words, up to $\mathscr{H}^1$-negligible sets they can be written as the countable union of Lipschitz images, that is there are Lipschitz functions $f_i : [0,1] \mapsto \mathbb{R}^d$ such that

$$\mathscr{H}^1\left(\Sigma \setminus \bigcup_{i\in\mathbb{N}} f_i([0,1])\right) = 0.$$

As such, this class of sets enjoy tangentiability properties $\mathscr{H}^1$ almost everywhere, see for instance [6, 9]. In other words, we know from the so called *blow-up Theorem* ([9, Thm. 10.2]) that for *a.e.* $x \in \Sigma$, it holds that

$$\frac{1}{r}\left(\Phi^{x,r}\right)_\sharp \mathscr{H}^1 \llcorner \Sigma = \mathscr{H}^1 \llcorner \left(\frac{\Sigma - x}{r}\right) \xrightarrow[r\to 0]{\star} \mathscr{H}^1 \llcorner T_x\Sigma, \text{ where } \Phi^{x,r} \overset{\text{def.}}{=} \frac{\mathrm{id} - x}{r}, \tag{2.4}$$

and $T_x\Sigma$ is a one-dimensional subspace of $\mathbb{R}^d$, which is called the *approximate tangent space of* Σ *at* x. This result holds for general $\mathscr{H}^k$-rectifiable sets, but a particularity of the 1-dimensional case is that we can use Gołąb's Theorem to prove the convergence of blow-ups in the Hausdorff and Kuratowski topologies as well.

Lemma 2.3. *Let* $\Sigma \subset \mathbb{R}^d$ *be closed and connected with* $\mathscr{H}^1(\Sigma) < +\infty$, *then for every* $x \in \Sigma$ *admitting an approximate tangent space* $T_x\Sigma$ *as in* (2.4), *and for all* $R > 0$ *it holds that*

$$\frac{\Sigma - x}{r} \cap \overline{B_R(0)} \xrightarrow[r\to 0^+]{d_H} T_x\Sigma \cap \overline{B_R(0)}, \tag{2.5}$$

as well as global convergence holds in the Kuratowski sense

$$\frac{\Sigma - x}{r} \xrightarrow[r\to 0^+]{K} T_x\Sigma.$$

In addition, for every $r >$ *it holds that*

$$d_H(\Sigma \cap B_r(x) - x, T_x\Sigma \cap B_r(0)) = r d_H\left(\frac{\Sigma - x}{r} \cap B_1, T_x\Sigma \cap B_1\right) = o(r). \tag{2.6}$$

Proof. First we take a rectifiability point $x \in \Sigma$ with tangent space $T_x\Sigma$, which we know to be $\mathscr{H}^1$ a.a. of Σ, so that (2.4) holds. Let T be the (Kuratowski) limit of a subsequence $\dfrac{\Sigma - x}{r_k}$. From (2.4) we have that $T_x\Sigma \subset T$. Thanks to Theorem 2.2, for almost all $R > 0$ it holds that

$$\mathscr{H}^1(T \cap B_R(0)) \le \liminf_{k\to\infty} \mathscr{H}^1\left(\frac{\Sigma - y}{r_k} \cap B_R(0)\right) = \mathscr{H}^1(T_y\Sigma \cap B_R(0)), \tag{2.7}$$

which shows $T \Delta T_x\Sigma$ is $\mathscr{H}^1$-negligible.

Notice that, if there is some $z \in T \setminus T_x\Sigma$, we may consider some ball $B_s(z)$ which does not intersect $T_x\Sigma$. Since T is the limit of connected sets, z must be path-connected in T to some point in $(B_s(z))^c$, so that $\mathscr{H}^1(T \cap B_s(z)) \ge s$. This contradicts (2.7). Hence, $T = T_x\Sigma$, and is independent of the subsequence, and we deduce the localized Hausdorff and the Kuratowski convergences.

To check (2.6), notice that from homogeneity of the distance in $\mathbb{R}^d$ it holds that

$$\frac{d_H((\Sigma - x) \cap B_r, T_x\Sigma \cap B_r)}{r} = d_H\left(\frac{\Sigma - x}{r} \cap B_1, T_x\Sigma \cap B_1\right)$$

and the RHS converges to zero as $r \to 0$ from the previous reasoning. $\square$

2.4. **Loops and tree structure.** We finally arrive at the central objects of the present work, which are loops from a connected set of finite length, or rather the absence of them. We start by properly defining what we mean by a loop.

Definition 2.4. We say that a set Γ is a *loop* whenever it is homeomorphic to $\mathbb{S}^1$. Any connected set Σ which contains no loops it is said to be a *tree*.

A point $x \in \Sigma$ is a *non-cut point of* Σ if $\Sigma \setminus \{x\}$ remains connected. Otherwise, x is called a *cut point*.

It turns out that $\mathscr{H}^1$ almost every point in a loop is a non-cut point. This is proved for instance in [13, Lemma 5.6] when the ambient space is a general metric space. In the following Lemma, we exploit the geometric structure of $\mathbb{R}^d$ to prove this result, while obtaining more information in the process.

Lemma 2.5. *Let $\Sigma \subset \mathbb{R}^d$ be a closed connected set with $\mathscr{H}^1(\Sigma) < +\infty$, consisting of more than one point and containing a loop Γ. Then $\mathscr{H}^1$-a.e. point $x \in \Gamma$ is such that for any $r > 0$ small enough, there exists $\bar{r} \in \left(\frac{r}{2}, r\right)$, such that $\Sigma \setminus B_{\bar{r}}(x)$ and $\Sigma \cap B_{\bar{r}}(x)$ are connected and*

$$\mathscr{H}^0(\Sigma \cap \partial B_{\bar{r}}(x)) = \mathscr{H}^0(\Gamma \cap \partial B_{\bar{r}}(x)) = 2.$$

In addition, it holds that $\mathscr{H}^1$-a.e. point of Γ is a non-cut point.

Proof. Let Γ be a loop of Σ, from the blow-up Theorem [9, Prop. 10.5], we know that $\mathscr{H}^1$-a.e. point of $\Sigma \cap \Gamma$ admits an approximate tangent plane such that

$$T_x\Sigma = T_x\Gamma.$$

Fix one such point x where the approximate tangents w.r.t. Σ and Γ coincide and let $\mathbb{R}\tau$ be the common tangent space. Given $r > 0$, it holds from the area formula and the blow-up Theorem that

$$\int_0^r \mathscr{H}^0(\partial B_s(x) \cap \Gamma)\mathrm{d}s \leq \int_0^r \mathscr{H}^0(\partial B_s(x) \cap \Sigma)\mathrm{d}s \leq \mathscr{H}^1(B_r(x) \cap \Sigma) = 2r + o(r). \tag{2.8}$$

In addition, from the Hausdorff convergence of the blow-ups from $\Sigma \cap B_r(x)$, Lemma 2.3, we can assume for n large enough that

$$\Sigma \cap B_r(x) \subset \left\{ z : \begin{array}{c} |\langle z - x, \tau\rangle| < r \\ \left|\langle z - x, \tau^\perp\rangle\right| < \frac{r}{100} \end{array} \right\}.$$

Since $\dfrac{\Gamma - x}{r}$ is a curve converging to the segment $\mathbb{R}\tau$, it must cross all the surfaces

$$\partial\left(B_s(0) \cap \{\pm\langle z, \tau\rangle > 0\}\right) \quad 0 < s < r,$$

so that $2 \leq \mathscr{H}^0(\Gamma \cap \partial B_s(x)) \leq \mathscr{H}^0(\Sigma \cap \partial B_s(x))$. As a result, from (2.8) we have that

$$0 \leq \frac{1}{r}\int_0^r \underbrace{\left(\mathscr{H}^0(\partial B_s(x) \cap \Gamma) - 2\right)}_{\geq 0} \mathrm{d}s \leq \frac{o(r)}{r}.$$

Hence, for r small enough, we can find

$$\bar{r} \in \left(\frac{r}{2}, r\right) \text{ such that } \mathscr{H}^0(\Sigma \cap \partial B_{\bar{r}}(x)) = \mathscr{H}^0(\Gamma \cap \partial B_{\bar{r}}(x)) = 2.$$

For such radius we have that $\partial B_{\bar{r}}(x) \cap \Sigma = \partial B_{\bar{r}}(x) \cap \Gamma = \{y_{1,n}, y_{2,n}\}$ and $\Gamma \setminus B_{\bar{r}}(x)$ is a path between $y_{1,n}$ and $y_{2,n}$.

It follows that both $\Sigma\cap B_{\bar r}(x)$ and $\Sigma\setminus B_{\bar r}(x)$ remain connected. Indeed, for the former, it suffices to notice that since $\mathscr{H}^0(\Gamma\cap B_{\bar r}(x))=2$, $\Gamma\cap B_{\bar r}(x)$ is homeomorphic to an arc of $\mathbb{S}^1$ and so it is connected, as continuous images of connected sets are connected. As a result, it must also hold that $\Sigma\cap B_{\bar r}(x)$ is connected since if it was not, there would a connected component Γ' that is disjoint fom $\Gamma\cap B_{\bar r}(x)$. But since $\Sigma\cap\partial B_{\bar r}(x)=\Gamma\cap\partial B_{\bar r}(x)$, Γ' would also be disjoint from $\Sigma\setminus B_{\bar r}(x)$, contradicting the connectedness of Σ.

To prove the connectedness of $\Sigma\setminus B_{\bar r}(x)$, consider $z_1,z_2\in\Sigma\setminus B_{\bar r}(x)$ and let $\gamma\subset\Sigma$ be a path between them. If $\gamma\subset\Sigma\setminus B_{\bar r}(x)$, there is nothing to prove, otherwise γ must contain either $y_{1,n}$ $y_{2,n}$, or both. If it contains only one of them, $\gamma\setminus B_{\bar r}(x)$ remains connected. In the case that it contains both, we can create a new path $\gamma\cup\Gamma\setminus B_{\bar r}(x)$ that must be connected, contained in $\Sigma\setminus B_{\bar r}(x)$ and has the points z_1,z_2. It follows that $\Sigma\setminus B_{\bar r}(x)$ is connected.

Let us show that x is a non-cut point. Indeed, for any $y_1,y_2\in\Sigma\setminus\{x\}$, use the previous construction to obtain a radius such that $\Sigma\setminus B_r(x)$ is connected and contains y_1,y_2. Therefore, we can find a path in $\Sigma\setminus\{x\}$ connecting them proving that $\{x\}$ is a non-cut point. ☐

As previously mentioned, Lemma 2.5 is a slight improvement over [4, Lemma 6.1] that is particularly useful to the localization arguments, since the latter provides a neighborhood D_n around a.e. non-cut point, but we have no information on the blowup of this set, complicating the implementation of the localization/blow-up argument. With the construction provided by Lemma 2.5, the limits of blow-up sequences are directly obtained via Lemma 2.3.

3. Absense of loops

In this section we fix $\nu_\star\in\mathscr{P}(\mathbb{R}^d)$, a minimizer of problem $(\overline{P}_\Lambda)$, along with its support Σ and set $\alpha\overset{\text{def.}}{=}\mathcal{L}(\nu_\star)$. We seek to perform the construction that will show that Σ is a tree. We recall the two cases described in Section 1.1 for which this will be shown:

Case 1: if ϱ_0 is a convex combination of Dirac masses, *i.e.*

$$\varrho_0=\mu_N=\sum_{i=1}^N a_i\delta_{x_i},\ \text{for}\ \sum_{i=1}^N a_i=1.$$

Case 2: ϱ_0 is absolutely continuous w.r.t. the Lebesgue measure with compact support and bounded density, *i.e.* $\varrho_0\in L^\infty(\Omega)$.

In the course of the proof we will need to transport part of the measure ϱ_0 with an arbitrary measurable selection of the projection operator

$$\Pi_\Sigma(x)=\operatorname*{argmin}_{y\in\Sigma}\frac{1}{2}|x-y|^2. \tag{3.1}$$

Therefore, we assume that

(3.2) there is a measurable selection π_Σ of (3.1) ϱ_0-a.e. uniquely defined.

This holds in

- case 1, since for each i we can choose $y_i\in\operatorname*{argmin}_{\Sigma}|x_i-y|^2$ and define $\pi_\Sigma(x_i)\overset{\text{def.}}{=}y_i$;

- case 2, since the projection map is Lebesgue-a.e. uniquely-defined.

3.1. **Loops are formed though projections.** In this paragraph we implement Step 1 of the localization/blow-up argument described in Section 1.1 by showing that loops are formed through projections onto the optimal network.

Proposition 3.1. *Suppose that* ϱ_0 *has a compact support and that* (3.2) *holds. Let* $\nu_\star$ *be a minimizer of* $(\overline{P}_\Lambda)$. *If* γ *is an optimal transportation plan between* ϱ_0 *and* $\nu_\star$ *and* $\Gamma \subset \Sigma$ *is a loop, then*

$$|x-y| = \operatorname{dist}(x,\Sigma) \textit{ for } \gamma\textit{-a.e. } (x,y) \in \mathbb{R}^d \times \Gamma.$$

Proof. Given $\eta > 0$, define the set

$$E_\eta \overset{\text{def.}}{=} \left\{ (x,y) \in \mathbb{R}^d \times \Gamma : \ |x-y|^p > \operatorname{dist}(x,\Sigma)^p + \eta \right\}$$

and consider the measure ν_η defined for a Borel set A as

$$\nu_\eta(A) \overset{\text{def.}}{=} \gamma(E_\eta \cap (\mathbb{R}^d \times A)).$$

From its construction, it follows that $\nu_\eta \le \nu_\star$. Therefore, to conclude it suffices to show that for any $\bar{y} \in \Sigma$, admitting an approximate tangent space $T_{\bar{y}}\Sigma = T_{\bar{y}}\Gamma$, it holds that

$$\theta_1(\nu_\eta, \bar{y}) = 0 \text{ for } \mathscr{H}^1\text{-a.e. } y \in \Gamma.$$

Let $(r_n)_{n\in\mathbb{N}}$ be an infinitesimal sequence obtained from Lemma 2.5 such that $\Sigma_n \overset{\text{def.}}{=} \Sigma \setminus B_{r_n}(\bar{y})$ remains connected. For n large enough, let us show that if

$$(x,y) \in E_\eta \cap (\mathbb{R}^d \times B_{r_n}(\bar{y})) \text{ then } \pi_\Sigma(x) \in \Sigma \setminus B_{r_n}(x).$$

Indeed, for such a pair (x,y) we have that

$$\begin{aligned}
\operatorname{dist}(x,\Sigma)^p + \eta \le |x-y|^p &\le \big(\operatorname{dist}(x,\Sigma) + |\pi_\Sigma(x) - y|\big)^p \\
&\le \operatorname{dist}(x,\Sigma)^p + p\big(\operatorname{dist}(x,\Sigma) + |y - \pi_\Sigma(x)|\big)^{p-1} |y - \pi_\Sigma(x)| \\
&\le \operatorname{dist}(x,\Sigma)^p + p\big(2\operatorname{diam}(\operatorname{supp}\varrho_0)\big)^{p-1} |y - \pi_\Sigma(x)|,
\end{aligned}$$

where the third inequality follows from the convexity of $t \mapsto |t|^p$. As a result, for n sufficiently large, we obtain that

$$2r_n < \frac{\eta}{p\big(2\operatorname{diam}(\operatorname{supp}\varrho_0)\big)^{p-1}} \le |y - \pi_\Sigma(x)|.$$

Since $y \in B_{r_n}(\bar{y})$, it must follow that $\pi_\Sigma(x) \in \Sigma \setminus B_{r_n}(\bar{y})$, for n large enough.

In the sequel, we write $B_{r_n} = B_{r_n}(\bar{y})$ to simplify notation, and we define an alternative transportation plan as follows

$$\gamma' \overset{\text{def.}}{=} \gamma \llcorner \mathbb{R}^d \times \Sigma_n + (\pi_0, \pi_\Sigma \circ \pi_0)_\sharp \gamma \llcorner E_\eta \cap \mathbb{R}^d \times B_{r_n} + (\pi_0, y_n)_\sharp \gamma \llcorner \mathbb{R}^d \times B_{r_n} \setminus E_\eta, \tag{3.3}$$

where π_0, π_1 denote the projections onto the first and second marginal, *i.e* $\pi_0(x,y) = x$, and $y_n \in \Sigma_n \cap \partial B_{r_n}(\bar{y})$. Its second marginal then defines a new competitor as

$$\nu' \overset{\text{def.}}{=} \nu_\star \llcorner \Sigma_n + \nu_\eta \llcorner B_{r_n} + \gamma\Big(\mathbb{R}^d \times B_{r_n} \setminus E_\eta\Big) \delta_{y_n}. \tag{3.4}$$

The first term preserves the transportation plan that does not concern $\Sigma \cap B_{r_n}$, the second projects onto Σ all the mass that is sent to $\Sigma \cap B_{r_n}$, and the last term sends all the mass whose projection is close to $\Sigma \cap B_{r_n}$ to the point y_n, creating a Dirac mass at y_n.

Since the mass on the second term of the transportation plan γ' in (3.3) is sent to Σ_n, it follows that $\operatorname{supp} \nu' = \Sigma_n$. But since this operation can only increase the density of $\nu_\star$ over Σ_n, we have that $\nu' \llcorner \Sigma_n \geq \nu_\star \llcorner \Sigma_n$ and it follows that

$$\mathcal{L}(\nu_\star) \geq \mathcal{L}(\nu'). \tag{3.5}$$

This construction yields

$$\begin{aligned} W_p^p(\varrho_0, \nu_\star) &= \int_{\mathbb{R}^d \times \Sigma_n} |x-y|^p \mathrm{d}\gamma + \int_{\mathbb{R}^d \times B_{r_n} \cap E_\eta} |x-y|^p \mathrm{d}\gamma + \int_{\mathbb{R}^d \times B_{r_n} \setminus E_\eta} |x-y|^p \mathrm{d}\gamma \\ &\geq \int_{\mathbb{R}^d \times \Sigma_n} |x-y|^p \mathrm{d}\gamma + \int_{\mathbb{R}^d \times B_{r_n} \cap E_\eta} \big(\operatorname{dist}(x,\Sigma)^p + \eta\big) \mathrm{d}\gamma \\ &+ \int_{\mathbb{R}^d \times B_{r_n} \setminus E_\eta} |x-y_n|^p \mathrm{d}\gamma - p \int_{\mathbb{R}^d \times B_{r_n} \setminus E_\eta} \underbrace{\big||x-y_n| - |x-y|\big|}_{\leq |y-y_n| \leq 2r_n} |x-y_n|^{p-1} \mathrm{d}\gamma \\ &\geq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p \mathrm{d}\gamma' + \eta \nu_\eta(B_{r_n}) - 2pr_n \int_{\mathbb{R}^d \times B_{r_n} \setminus E_\eta} |x-y_n|^{p-1} \mathrm{d}\gamma, \end{aligned}$$

so that from the minimality of $\nu_\star$ and (3.5), the previous estimate gives

$$\frac{\nu_\eta(B_{r_n}(\bar{y}))}{2r_n} \leq \frac{p}{\eta} \int_{\mathbb{R}^d \times B_{r_n} \setminus E_\eta} |x-y_n|^{p-1} \mathrm{d}\gamma \xrightarrow[n \to \infty]{} 0.$$

We conclude that for all $\bar{y}$ that is a rectifiability point of Γ, it holds that $\theta_1(\nu_\eta, \bar{y}) = 0$, and the result follows. □

3.2. **Localizations and blow-up.** Since we know from Prop 3.1 that loops are formed though projections, we can perform Step 2 from Section 1.1. That is, we chose a suitable point to perform localizations.

As the proof is by contradiction, we first assume that Σ contains a loop Γ. We consider

$$y_0 \in \Gamma, \text{ is a noncut point such that } T_{y_0}\Sigma = T_{y_0}\Gamma, \tag{3.6}$$

which can be done since, $\mathscr{H}^1$-a.e., the approximate tangent spaces to Σ and Γ coincide. In Case 1, where ϱ_0 is atomic, we make the additional assumption

$$y_0 \neq x_i, \text{ for all } i = 1, \dots, N. \tag{3.7}$$

Next, let $(r_n)_{n \in \mathbb{N}}$ be a sequence of radii obtained from Lemma 2.5, and we introduce the following notation

$$\Sigma_{y_0, r_n} \overset{\text{def.}}{=} \Sigma \cap B_{r_n}(y_0), \ \Sigma_n \overset{\text{def.}}{=} \Sigma \setminus \Sigma_{y_0, r_n}, \tag{3.8}$$

so that from Lemma 2.5 it holds that

$$\Sigma_{y_0, r_n} \text{ and } \Sigma_n \text{ are connected and } r_n \to 0. \tag{3.9}$$

In the sequel, we will focus our attention into the following sequence of localized measures

$$\nu_n \overset{\text{def.}}{=} \nu_\star \llcorner \Sigma_{y_0, r_n}.$$

From the optimality of $\nu_\star$, this sequence minimize a family of localized variational problems consisting of the transportation of "the portion of ϱ_0 that is sent to ν_n", namely

$$\varrho_n \overset{\text{def.}}{=} (\pi_0)_\sharp\left(\gamma \llcorner \Omega \times \Sigma_{y_0,r_n}\right).$$

In Case 2, we can equivalently write $\varrho_n = \varrho_0 \llcorner T^{-1}(\Sigma_{y_0,r_n})$, where T corresponds to the optimal transportation map from ϱ_0 to $\nu_\star$.

Afterwards, we define a blow-up of this sequence of problems and extract a limit. But to prevent the measure ϱ_n from losing mass at infinity in the blow-up step, as in [5], we let ϱ_n follow a constant speed geodesic in the Wasserstein space almost until it reaches ν_n, defined as follows: if γ_n is an optimal transportation plan between ϱ_n and ν_n, we are interested in the following geodesic interpolation between them

$$\sigma_n \overset{\text{def.}}{=} \left(\pi_{r_n}\right)_\sharp \gamma_n \text{ where } \pi_{r_n} \overset{\text{def.}}{=} r_n \pi_0 + (1-r_n)\pi_1. \tag{3.10}$$

The reader is referred to [15, Thm. 5.27] for a proof of the fact that the above interpolation indeed yields geodesics for the W_p distance.

With these elements we obtain the following result, whose proof is included in Appendix A for completeness since it is a minor variant of the results found in [5]. But as we are interested in making variations that will "open" the loop Γ, to simplify notation we define the following class of sets

$$\mathcal{A}_2 \overset{\text{def.}}{=} \left\{\Sigma' \subset \overline{B_1(0)} : \Sigma \text{ has at most 2 connected components}\right\}. \tag{3.11}$$

Lemma 3.2. *The localized measure ν_n solves the following minimization problem*

$$\min \left\{ W_p^p(\sigma_n, \nu') : \begin{array}{c} \textit{there is } \Sigma' \in \mathcal{A}_2 \textit{ such that} \\ \nu' \in \mathcal{M}_+(\Sigma'),\ \nu' \geq \alpha^{-1}\mathcal{H}^1 \llcorner \Sigma', \\ \Sigma_n \cup \Sigma' \textit{ is connected,} \\ \nu'(\overline{B_1(0)}) = \nu_\star\left(\Sigma_{y_0,r_n}\right) \end{array} \right\}. \tag{3.12}$$

In the sequel, recalling the definition of the blow-up operator $\Phi^{y_0,r} = \frac{\mathrm{id}-y_0}{r}$ from (2.4) in Section 2.3, notice that for any given measures μ, ν it holds that

$$W_p^p\left(\frac{1}{r}(\Phi^{y_0,r})_\sharp \mu, \frac{1}{r}(\Phi^{y_0,r})_\sharp \nu\right) = \frac{1}{r^{p+1}} W_p^p\left(\mu, \nu\right). \tag{3.13}$$

We are particularly interested in the sequences of blow-ups of the measures σ_n and ν_n:

$$\bar{\sigma}_n \overset{\text{def.}}{=} \frac{1}{r_n}(\Phi^{y_0,r_n})_\sharp \sigma_n, \quad \bar{\nu}_n \overset{\text{def.}}{=} \frac{1}{r_n}(\Phi^{y_0,r_n})_\sharp \nu_n, \tag{3.14}$$

since we already know from Lemma 3.2 that they will inherit some optimality property.

From Lemma 3.2 and (3.13), each element from the sequence $(\bar{\nu}_n)_{n\in\mathbb{N}}$ is almost an minimizer of a sequence of functionals $(F_n)_{n\in\mathbb{N}}$, see Lemma 3.4 below, defined as

$$F_n(\nu') \overset{\text{def.}}{=} \begin{cases} W_p^p\left(\bar{\sigma}_n, \nu'\right), & \begin{array}{c} \text{there is } \Sigma' \in \mathcal{A}_2 \text{ such that} \\ \nu' \in \mathscr{M}_+(\Sigma'),\ \nu' \geq \alpha^{-1}\mathscr{H}^1 \llcorner \Sigma', \\ \left(\dfrac{\Sigma_n - y_0}{r_n}\right) \cup \Sigma' \text{ is connected}, \\ \nu'(\overline{B_1(0)}) = \dfrac{\nu_\star\left(\Sigma_{y_0, r_n}\right)}{r_n}, \end{array} \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.15}$$

Now, recall that from the blow-up properties of Σ, if follows that

$$\frac{\Sigma_{y_0,r_n} - y_0}{r_n} \xrightarrow[n\to\infty]{d_H} T_{y_0}\Sigma \cap \overline{B_1(0)}.$$

We can also extract a subsequence for the convergence of the measures, so that it holds that

$$\bar{\sigma}_n \xrightarrow[n\to\infty]{\star} \bar{\sigma}, \quad \bar{\nu}_n \xrightarrow[n\to\infty]{\star} \bar{\nu}. \tag{3.16}$$

This motivates the following limit problem, which is minimized by $\bar{\nu}$ as we shall prove later,

$$F(\nu') \overset{\text{def.}}{=} \begin{cases} W_p^p\left(\bar{\sigma}, \nu'\right), & \begin{array}{c} \text{there exists } \Sigma' \in \mathcal{A}_2 \text{ such that} \\ \nu' \in \mathscr{M}_+(\Sigma'),\ \nu' \geq \alpha^{-1}\mathscr{H}^1 \llcorner \Sigma', \\ T_{y_0}\Sigma \cap \partial B_1(0) \subset \Sigma', \\ \nu'(\overline{B_1(0)}) = 2\theta_1(\nu_\star, y_0), \end{array} \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.17}$$

Step 3 from 1.1 consists of defining the functionals F_n above and show that they Γ-converge to F. This is done in the following Theorem, whose proof is also left to the Appendix A.

Theorem 3.3. *The family $(F_n)_{n\in\mathbb{N}}$ converges to F in the sense of Γ-convergence, for the topology of weak-$\star$ convergence of Radon measures.*

In Step 4, we transfer a lot of information about the minimization of F_n to the minimization of F, by means of the Γ-convergence result and the fact that the optimal transportation in the definition of F_n is almost achieved via projections. In fact, only the transportation onto $\Gamma \cap B_{r_n}(y_0)$ is given by projections, and there might be some mass in the set $(\Sigma \setminus \Gamma) \cap B_{r_n}(y_0)$, but since Σ and Γ have the same approximate tangent space at y_0, this contribution vanishes as $n \to \infty$, and the limit inherits the projection properties from the loop Γ. This discussion is formalized below.

Lemma 3.4. *The following assertions are true:*

(i) We have $\bar{\nu} = 2\theta_1(\nu_\star, y_0)\mathscr{H}^1 \llcorner T_{y_0}\Sigma \cap B_1(0)$ and it is a minimizer of F;

(ii) The following assertions about $\bar{\sigma}$ hold:

<u>Case 1:</u> Define the quantity

$$0 < L \overset{\text{def.}}{=} \min_{i=1,\dots,N} |y_0 - x_i|.$$

Then we have that $\operatorname{supp} \bar{\sigma} \subset \{\operatorname{dist}(\cdot, T_{y_0}\Sigma) \geq L\}$*;*

Case 2: $\bar\sigma(T_{y_0}\Sigma\cap\overline{B_1(0)})=0$.

(iii) the optimal transportation from $\bar\sigma$ to $\bar\nu$ is attained by the projection map onto $T_{y_0}\Sigma$.

Proof. Starting with item (i), recall that

$$\bar\nu_n=\frac{1}{r_n}\left(\Phi^{y_0,r_n}\right)_\sharp\nu_n,$$

where ν_n is a minimizer of (3.12) thanks to Lemma 3.2. As a result, Σ_{y_0,r_n} satisfies the restrictions of (3.12). As a result, the set $\dfrac{\Sigma_{y_0,r_n}-y_0}{r_n}$ satisfy all the restrictions of F_n for $\bar\nu_n$. On the other hand, given any ϱ satisfying the restrictions of F_n with a set Σ' yields $\nu'=r_n(\Phi^{y_0,r_n})^{-1}_\sharp\varrho$ admissible for (3.12) with the set $y_0+r_n\Sigma'$. Indeed, the only property that requires checking is that $\nu'\geq\alpha^{-1}\mathscr{H}^1\llcorner(y_0+r_n\Sigma')$, which follows directly from the area formula since, for any continuous $\phi\geq 0$, we have

$$\begin{aligned}\int\phi\mathrm{d}\nu&=r_n\int\phi(y_0+r_nx)\mathrm{d}\varrho(x)\geq\alpha^{-1}r_n\int_{\Sigma'}\phi(y_0+r_nx)\mathrm{d}\mathscr{H}^1(x)\\&=\alpha^{-1}r_n\int_{y_0+r_n\Sigma'}\phi\mathrm{d}\mathscr{H}^1.\end{aligned}$$

As a result, using identity (3.13), it follows that

$$\begin{aligned}W_p^p(\bar\sigma_n,\bar\nu_n)&=\frac{1}{r_n^{p+1}}W_p^p(\sigma_n,\nu_n)\leq\frac{1}{r_n^{p+1}}W_p^p(\sigma_n,\nu)\\&\leq W_p^p(\bar\sigma_n,\varrho).\end{aligned}$$

Showing that $\bar\nu_n$ is a sequence of minimizers, so that the minimality of $\bar\nu$ follows from the fundamental properties of Γ convergence.

Moving on to item (ii), the first case follows directly from the fact that ϱ_0 is atomic. To prove the second case, first we recall that since ϱ_0 is absolutely continuous, its optimal transportation is uniquely attained by a map T, and we can write $\sigma_n=T_{r_n\sharp}\varrho_n$, with $T_{r_n}\overset{\text{def.}}{=}r_n\mathrm{id}+(1-r_n)T$ and ϱ_n-a.e. $T=\pi_\Sigma$, thanks to Prop. 3.1. Next, we define the open set

$$C_\delta\overset{\text{def.}}{=}\left\{x:\operatorname{dist}\left(x,T_{y_0}\Sigma\right)<\delta\right\},$$

so that for all $\delta>0$ we have that

$$\bar\sigma(C_\delta)\leq\liminf_{\delta\to 0}\bar\sigma_n(C_\delta),\tag{3.18}$$

where by definition we have that

$$\bar\sigma_n(C_\delta)=r_n^{-1}\varrho_n\left(T_{r_n}{}^{-1}(y_0+r_nC_\delta)\right).$$

Hence, let us study the set $T_{r_n}{}^{-1}(y_0+r_nC_\delta)$. Consider a pair (x,y) such that $y\in y_0+r_nC_\delta$, $x\in\operatorname{supp}\varrho_n$ and

$$y=T_{r_n}(x)=r_nx+(1-r_n)T(x).\tag{3.19}$$

Since for ϱ_n-a.e. x, the map T behaves as a projection onto Σ, and the map T_{r_n} is an interpolation between the identity and the projection onto Σ, it follows that $T(x) = \pi_\Sigma(x) = \pi_\Sigma(y)$. In addition, rearranging the terms in (3.19) we obtain

$$r_n(x - T(x)) = y - T(x) = r_n\left(\frac{y-y_0}{r_n} - \frac{\pi_\Sigma(y)-y_0}{r_n}\right)$$

so that recalling that $y \in y_0 + r_n C_\delta$, it holds that

$$\begin{aligned}\operatorname{dist}(x, \Sigma_{y_0,r_n}) = |x - T(x)| &= \left|\frac{y-y_0}{r_n} - \frac{\pi_\Sigma(y)-y_0}{r_n}\right| \\ &= \operatorname{dist}\left(\frac{y-y_0}{r_n}, \frac{\Sigma_{y_0,r_n}-y_0}{r_n}\right) \\ &= \operatorname{dist}\left(\frac{y-y_0}{r_n}, T_{y_0}\Sigma \cap B_1(0)\right) + o_{n\to\infty}(1),\end{aligned}$$

where the last equality follows from the equivalence of convergence in the Hausdorff distance and uniform convergence of the distance functions. We conclude that for n sufficiently large $\operatorname{dist}(x, \Sigma_{y_0,r_n}) \le 2\delta$, so that

$$\operatorname{supp}\varrho_n \cap T_{r_n}^{-1}\left(y_0 + r_n C_\delta\right) \subseteq \operatorname{supp}\varrho_n \cap \left\{\operatorname{dist}\left(\cdot, \Sigma_{y_0,r_n}\right) \le 2\delta\right\}.$$

Returning to (3.18) with this new inclusion we conclude that

$$\bar\sigma(C_\delta) \le \liminf_{\delta\to 0} r_n^{-1}\varrho_0\left(\left\{\operatorname{dist}\left(\cdot, \Sigma_{y_0,r_n}\right) \le \delta\right\}\right).$$

Now assume by contradiction that there is an $\varepsilon > 0$ such that for all $\delta > 0$ the $\liminf$ on the RHS above is greater than ε. For any fixed δ, up to considering a subsequence that attains the $\liminf$, it would hold that for n large enough

$$\frac{\varepsilon r_n}{2} \le \varrho_n\left(\left\{\operatorname{dist}\left(\cdot, \Sigma_{y_0,r_n}\right) \le \delta\right\}\right), \tag{3.20}$$

so let us estimate this volume on the right-hand side, we claim that

$$\varrho_n\left(\left\{\operatorname{dist}\left(\cdot, \Sigma_{y_0,r_n}\right) \le \delta\right\}\right) \le \left\|\varrho_0\right\|_\infty \omega_{d-1}\mathscr{H}^1(\Sigma_{y_0,r_n})\delta^{d-1} + o(r_n). \tag{3.21}$$

This estimate will be proven with a slight refinement of the induction strategy from [10, Lemma 4.2].

First recall that by the construction from Lemma 2.5, both Σ_{y_0,r_n} and Σ_n are connected and we have that

$$\Sigma_{y_0,r_n} \cap \partial B_{r_n}(y_0) = \{y_{n,1}, y_{n,2}\}\Sigma_n \cap \partial B_{r_n}(y_0).$$

In particular, Σ_{y_0,r_n} is 1-rectifiable and can be covered by countably many connected sets $\left(\gamma_{n,i}\right)_{i\in\mathbb{N}}$. We assume without loss of generality that:

- $\gamma_{n,1} \subset \Gamma$;
- $\gamma_{n,1}$ contains the two points of Σ_{y_0,r_n} on the boundary $\partial B_{r_n}(y_0)$;
- and as a consequence $\mathscr{H}^1(\gamma_{n,1}) \ge 2r_n$.

In addition, we can assume that the remaining sets $\gamma_{n,i}$ are piece-wise disjoint and for all $i \ge 2$ we have that

$$\mathscr{H}^1(\gamma_{n,i}) \le \mathscr{H}^1(\Sigma_{y_0,r_n} \setminus \Gamma) = o(r_n),$$

where the last equality comes from the blow-up theorem and the fact that y_0 is a flat point of both Σ and Γ.

First we estimate the volume of the points at distance at most δ to $\gamma_{n,1}$. Indeed, we can decompose the set

$$A(\gamma_{n,1},\delta) \overset{\text{def.}}{=} \left\{\operatorname{dist}(\cdot,\gamma_{n,1}) \leq \delta\right\} \leq C(\gamma_{n,1},\delta) + H(\gamma_{n,1},\delta),$$

where $C((\gamma_{n,1},\delta))$ is a tubular region around $\gamma_{n,1}$ and $H(\gamma_{n,1},\delta)$ is a union of two hemispheres centered at its end-points. For the tubular region we have the bound

$$\varrho_n\left(C(\gamma_{n,1},\delta)\right) \leq \left\|\varrho_0\right\|_\infty \mathscr{H}^1(\gamma_{n,1})\omega_{d-1}\delta^{d-1}.$$

On the other hand, for the two hemispheres we have that

$$\varrho_n\left(H(\gamma_{n,1},\delta)\right) = o(r_n),$$

since either they are at minimal distance to Σ outside of B_{r_n}, hence not in the support of ϱ_n, or their projection onto Σ is contained in $\{y_{n,1}, y_{n,2}\} \cup \Sigma_{y_0,r_n} \setminus \Gamma$. Hence

$$\varrho_n\left(H(\gamma_{n,1},\delta)\right) \leq \nu_n\left(\{y_{n,1}, y_{n,2}\} \cup \Sigma_{y_0,r_n} \setminus \Gamma\right) = o(r_n),$$

and we have proven the first step induction towards (3.21).

To finish the proof define

$$C_k \overset{\text{def.}}{=} \bigcup_{i=1}^{k} \gamma_{n,i},$$

assume that (3.21) holds with Σ_{y_0,r_n} replaced by C_k and let us show that it holds for C_{k+1}. In this case, we have that

$$\begin{aligned}
\varrho_n(A(C_{k+1},\delta)) &= \varrho_n(A(C_k,\delta)) + \varrho_n(A(\gamma_{n,k+1},\delta) \setminus A(C_k,\delta)) \\
&\leq \left\|\varrho_0\right\|_\infty \left[\omega_{d-1}\mathscr{H}^1(C_k)\delta^{d-1} + |A(\gamma_{n,k+1},\delta) \setminus A(C_k,\delta)|\right] + o(r_n).
\end{aligned}$$

Hence, let us estimate $|A(\gamma_{n,k+1},\delta) \setminus A(C_k,\delta)|$. Once again, $A(\gamma_{n,k+1},\delta)$ will have one tubular region and two hemispheres, but since $\gamma_{n,k+1}$ touches C_k, we can remove at least one ball of volume $\omega_d\delta^d$, which makes up for the two hemispheres. This way we have that

$$\begin{aligned}
\varrho_n(A(C_{k+1},\delta)) &\leq \left\|\varrho_0\right\|_\infty \omega_{d-1}\delta^{d-1}\left[\mathscr{H}^1(C_k) + \mathscr{H}^1(\gamma_{n,k+1})\right] + o(r_n) \\
&= \left\|\varrho_0\right\|_\infty \omega_{d-1}\delta^{d-1}\mathscr{H}^1(C_{k+1}) + o(r_n).
\end{aligned}$$

As a result, this estimate holds for every $k \in \mathbb{N}$, and since by construction $\mathscr{H}^1(C_k) \to \mathscr{H}^1(\Sigma_{y_0,r_n})$, we obtain the bound (3.21).

Going back to (3.20), for all δ and r_n sufficiently small we would have that

$$\varepsilon/2 \leq \left\|\varrho_0\right\|_\infty \frac{\mathscr{H}^1\left(\Sigma_{y_0,r_n}\right)}{r_n}\omega_{d-1}\delta^{d-1} + \frac{o(r_n)}{r_n} \xrightarrow[\substack{\delta\to 0\\ n\to+\infty}]{} 0,$$

which is a contradiction. We conclude that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $\bar{\sigma}(C_\delta) \leq \varepsilon$, implying that $\bar{\sigma}(T_{y_0}\Sigma \cap \overline{B_1(0)}) = 0$.

Finally, to prove item (iii), recall the sequences σ_n and $\nu_\star \llcorner B_{r_n}$, and let γ_n be the optimal transportation plan between them. From Prop. 3.1, it follows that

$$\operatorname{supp} \gamma_n \llcorner \mathbb{R}^d \times \Gamma \subset \operatorname{graph}(\Pi_\Sigma).$$

Since $\bar\sigma_n, \bar\nu_n$ are generated by the push-forward of σ_n and $\nu_\star \llcorner B_{r_n}$ by Φ^{y_0,r_n}, the optimal transportation between them in given by the plan

$$\bar\gamma_n \overset{\text{def.}}{=} \frac{1}{r_n}(\Phi^{(y_0,y_0),r_n})_\sharp \gamma_n, \text{ so that } \operatorname{supp}\left(\bar\gamma_n \llcorner \left(\mathbb{R}^d \times \frac{\Gamma - y_0}{r_n}\right)\right) \subset \operatorname{graph}\left(\Pi_{\frac{\Sigma - y_0}{r_n}}\right).$$

If Σ_{y_0,r_n} was entirely contained in Γ, the proof would be strictly the same as in the analogous result from [5]. Here this is not the case, but the set part of Σ_{y_0,r_n} where the projection property might fail is small since $\mathscr{H}^1(\Sigma_{y_0,r_n} \setminus \Gamma) = o(r_n)$.

Up to a subsequence $\bar\gamma_n$ converges to some $\bar\gamma$, which, by the stability of optimal transportation plans, also transports $\bar\sigma$ to $\bar\nu$ optimally, let us show that $\operatorname{supp}\bar\gamma \subset \operatorname{graph}\left(\Pi_{T_{y_0}\Sigma}\right)$. Notice that for any $A \subset \mathbb{R}^d$, we have that

$$\bar\gamma_n\left(A \times \left(\frac{\Sigma_{y_0,r_n} \setminus \Gamma - y_0}{r_n}\right)\right) \le \frac{1}{r_n}\nu_\star\left(\Sigma_{y_0,r_n} \setminus \Gamma\right) = \frac{o(r_n)}{r_n} \xrightarrow[n\to\infty]{} 0,$$

since $\theta_1(\nu_\star, y_0) < +\infty$ and the tangent spaces of Σ and Γ coincide at y_0, from (3.6).

As a result, given $(x,p) \in \operatorname{supp}\bar\gamma$, there is an open ball B centered at (x,p) such that

$$0 < \bar\gamma(B) \le \liminf_{n\to\infty} \bar\gamma_n(B) = \liminf_{n\to\infty} \bar\gamma_n\left(B \cap \left(\mathbb{R}^d \times \frac{\Gamma - y_0}{r_n}\right)\right).$$

In particular, we can find $\operatorname{supp}\bar\gamma_n \llcorner \left(\mathbb{R}^d \times \frac{\Gamma - y_0}{r_n}\right) \ni (x_n, p_n) \xrightarrow[n\to\infty]{} (x,p)$. So it holds that

$$|x - p| = \lim_{n\to\infty} |x_n - p_n| = \lim_{n\to\infty} \operatorname{dist}\left(x_n, \frac{\Sigma - y_0}{r_n}\right) = \operatorname{dist}(x, T_{y_0}\Sigma),$$

where the last equality comes from the point-wise convergence of the distance functions from Kuratowski convergence of blow-ups from Lemma 2.3.

□

3.3. **Better competitor and absence of loops.** We now implement Step 5 from Section 1.1 obtaining a contradiction to the fact that the optimal set Σ contains a loop. Let us recall the construction done so far; if Σ the support of an optimal measure for $(\overline{P}_\Lambda)$ which contains a loop Γ, we choose a suitable flat non-cut point $y_0 \in \Gamma$, as in (3.19). Then we can perform the localizations around y_0 from the previous subsection and obtain the measures $\bar\sigma$ and $\bar\nu$, as in (3.16). From Lemma 3.4, the latter is a minimizer of the functional F defined in (3.17) and

$$\bar\nu = \theta\mathscr{H}^1 \llcorner T_{y_0}\Sigma \in \operatorname{argmin} F, \text{ where } \theta = 2\theta_1(\nu_\star, y_0).$$

As the optimal transportation from $\bar\sigma$ to $\bar\nu$ is attained by the projection map onto $T_{y_0}\Sigma$, we use a refined version of the argument done in [5, Lemma 6.3] to construct a strictly better competitor to F. The further complexity of this case stems from the fact that we must remove all the mass of a small segment and create an advantageous structure, see Figure 3. This construction will then contradict the existence of loops, so that any optimal Σ must be a tree.

Theorem 3.5. *Let ϱ_0 be as in Case 1 or Case 2. Then any solution Σ to the problem (P_Λ) is a tree, in the sense that it does not contain homeomorphic images of $\mathbb{S}^1$.*

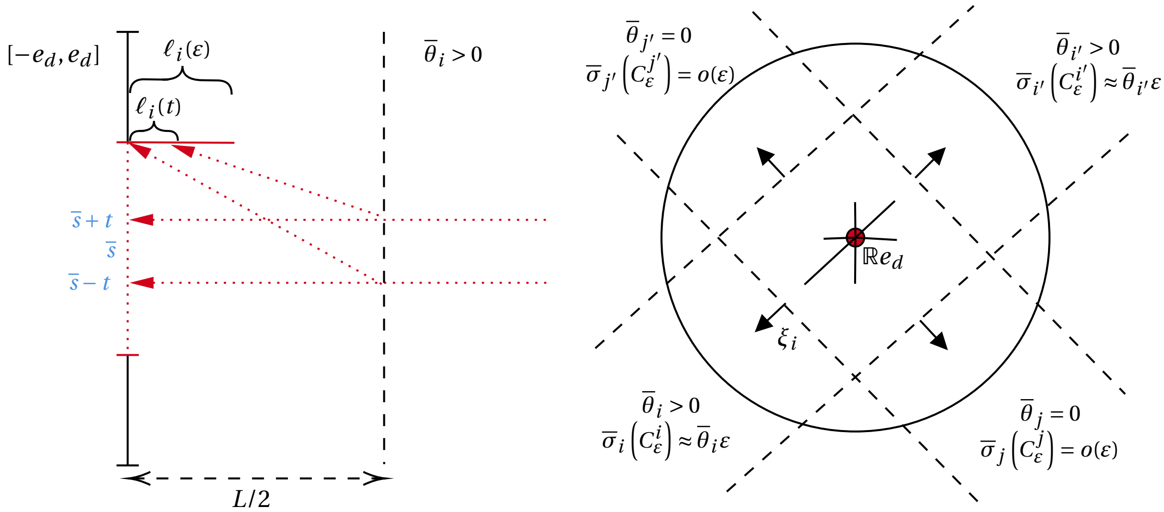

FIGURE 3. Construction of a better competitor in Thm. 3.5. On the right, the partition of the space into sections. For sections i, i' such that $\bar\theta_i, \bar\theta_{i'} > 0$ we add a segment in their direction. For $\bar\theta_j, \bar\theta_{j'} = 0$ we construct a Dirac mass. On the cases of positive density we have a gain of order ε^2 in transportation cost, for zero density we lose $o(\varepsilon^2)$. On the left the transportation strategy of each section of the partitioned space.

Proof. Suppose by contradiction that Σ is optimal and contains a loop, and let y_0 be a flat non-cut point inside this loop, chosen as in (3.19). Up to a rotation, we may assume that $T_{y_0}\Sigma = \mathbb{R}^d e_d$, where $(e_i)_{i=1}^d$ is a basis of $\mathbb{R}^d$. We will start with a simpler construction for Case 1; and then use it as a building block for the second one.

Case 1: Recall that $\operatorname{supp}\bar\sigma \subset \left\{x = (x', x_d) \in \mathbb{R}^d : |x'| > L,\ |x_d| \le 1\right\}$, as shown in item (i) of Lemma 3.4, so we can cover its support with finitely many sets $(E_i)_{i=1}^N$ defined as:

$$E_i \overset{\text{def.}}{=} \left\{x = (x', x_d) \in \mathbb{R}^d : \langle \xi_i, x\rangle > L/2,\ |x_d| \le 1\right\}$$

where $\xi_i \in \mathbb{S}^{d-1} \cap [e_d]^\perp$ are unit vectors and N depends only on the dimension. We then define a disjoint family

$$F_1 = E_1, \quad F_{i+1} = E_{i+1} \setminus \bigcup_{j=1}^{i} F_i \text{ for } i \ge 1$$

and decompose our measures $\bar\sigma$ and $\bar\nu$ as

$$\bar\sigma = \sum_{i=1}^N \bar\sigma_i,\ \bar\nu = \sum_{i=1}^N \bar\nu_i \text{ where } \bar\sigma_i \overset{\text{def.}}{=} \bar\sigma \llcorner F_i \text{ and } \bar\nu_i \overset{\text{def.}}{=} (\mathsf{proj}_d)_\sharp \bar\sigma_i,$$

where $\mathsf{proj}_d : x \mapsto x_d e_d$ is the projection onto the vertical axis. By Besicovitch's differentiation theorem, $\bar\nu_i = \theta_i \mathscr{H}^1 \llcorner [-e_d, e_d]$, where $\theta_i(s) = \theta_i(s e_d) \ge 0$ sum up to a positive constant

$$\sum_{i=1}^N \theta_i(s) = \theta > 0.$$

In the sequel, introduce the notation: $\mathbb{R}^d \ni x = (x_i, x_i'', x_d)$ where $x_i = \langle \xi_i, x\rangle$ is the component of x parallel to ξ_i and $x_i'' \in [\xi_i, e_d]^\perp$. Defining the sets

$$C_t^i \overset{\text{def.}}{=} F_i \cap \{x \in \mathbb{R}^d : |x_d - \bar{s}| \le t\} \subset \left\{ x = (x_i, x_i'', x_d) : \begin{matrix} x_i > L/2, \\ |x_d - \bar{s}| \le t \end{matrix} \right\},$$

and letting $\bar{s} \in (-1,1)$ be a common Lebesgue point of all θ_i, $i = 1, \dots, N$, it follows from the fact that $(\mathrm{proj}_d)_\sharp \bar{\sigma}_i = \theta_i \mathscr{H}^1 \llcorner [-e_d, e_d]$ that, for every $i = 1, \dots, N$

$$\frac{\bar{\sigma}_i(C_\varepsilon^i)}{2\varepsilon} = \frac{1}{2\varepsilon} \int_{\bar{s}-\varepsilon}^{\bar{s}+\varepsilon} \theta_i(t)\,\mathrm{d}t \xrightarrow[\varepsilon \to 0]{} \theta_i(\bar{s}). \tag{3.22}$$

Consider now the two subfamilies of indexes

$$I_1 \overset{\text{def.}}{=} \{i : \ \theta_i(\bar{s}) > 0\}, \quad I_2 \overset{\text{def.}}{=} \{i : \ \theta_i(\bar{s}) = 0\}. \tag{3.23}$$

In particular, for each $i \in I_1$, there is a constant $\bar{\theta}_i > 0$ and $\varepsilon > 0$ such that for $t < \varepsilon$ we have

$$\frac{1}{\bar{\theta}_i} \le \frac{\bar{\sigma}_i(C_t^i)}{t} \le \bar{\theta}_i. \tag{3.24}$$

Now let us exploit the fact that, from Lemma 3.4 the optimal transport is given by projections to propose a new transportation map, sending the mass in C_ε^i to a segment pointing towards ξ_i:

$$\bar{T}(x) \overset{\text{def.}}{=} \begin{cases} \ell_i(|x_d - \bar{s}|)\xi_i + (\bar{s}+\varepsilon)e_d, & \text{if } x \in C_\varepsilon^i \text{ and } i \in I_1, \\ (\bar{s}+\varepsilon)e_d, & \text{if } x \in C_\varepsilon^i \text{ and } i \in I_2, \\ \mathrm{proj}_d(x), & \text{otherwise}, \end{cases}$$

where $\ell_i : [0,\varepsilon] \to \mathbb{R}_+$ is defined via the conservation of mass relation

$$\ell_i(t) = \alpha \bar{\sigma}_i(C_t^i). \tag{3.25}$$

In other words, the mass that was sent to the vertical segment $[\bar{s}-\varepsilon', \bar{s}+\varepsilon']e_d$ is now used to form the horizontal segments

$$L_i \overset{\text{def.}}{=} (\bar{s}+\varepsilon)e_d + [0, \ell_i(\varepsilon)]\xi_i,$$

for each $i \in I_1$. The mass corresponding to the remaining indexes form a Dirac measure concentrated in $(\bar{s}+\varepsilon)e_d$, but with a mass of order $o(\varepsilon)$.

Thanks to (3.25), the map $\bar{T}$ sends $\bar{\sigma}_i \llcorner C_\varepsilon^i$ to the measure $\alpha^{-1}\mathscr{H}^1 \llcorner L_i$, hence the transported measure $\bar{T}_\sharp \bar{\sigma}$ satisfies the constraints in the definition (3.17) of the limiting functional F, since the newly added structure, given by

$$\Sigma' = \bigcup_{i \in I_1} L_i,$$

is a connected set. As a result, one has that $F(\bar{T}_\sharp \bar{\sigma}) < +\infty$.

So for $i \in I_1$ and $x \in C_\varepsilon^i$, recalling the notation $x = (x_i, x_i'', x_d)$, we have that

$$\begin{aligned} |x - \mathrm{proj}_d(x)|^2 - |x - \bar{T}(x)|^2 &= x_i^2 + |x_i''|^2 - (x_i - \ell_i(|x_d - \bar{s}|))^2 - |x_i''|^2 - (x_d - \bar{s} - \varepsilon)^2 \\ &= 2x_i \ell_i(|x_d - \bar{s}|) - \ell_i(|x_d - \bar{s}|)^2 - (x_d - \bar{s})^2 + 2\varepsilon|x_d - \bar{s}| - \varepsilon^2 \\ &\ge 2\left(\frac{L}{\alpha\bar{\theta}_i} + \varepsilon\right)|x_d - \bar{s}| - \left(1 + (\alpha\bar{\theta}_i)^2\right)|x_d - \bar{s}|^2 - \varepsilon^2 \\ &\ge \frac{2L}{\alpha\bar{\theta}_i}|x_d - \bar{s}| - \left(2 + (\alpha\bar{\theta}_i)^2\right)\varepsilon^2, \end{aligned}$$

This is a qualitative estimate on the difference of the squared distance, to extend it to the p-power, we use that for any $a, b > 0$

$$a^{p/2} - b^{p/2} = \frac{p}{2} b^{\frac{p}{2}-1}(a-b) + o(a-b), \tag{3.26}$$

so that since $|x_d - \bar{s}| < \varepsilon$ for $x \in C^i_\varepsilon$ and $|x - \bar{T}(x)| > \frac{1}{2}$, taking $a = |x - \mathrm{proj}_d(x)|^2$ and $b = |x - \bar{T}(x)|^2$, we obtain for some constant C_p that

$$\begin{aligned} |x - \mathrm{proj}_d(x)|^p - |x - \bar{T}(x)|^p &\geq C_p \left(|x - \mathrm{proj}_d(x)|^2 - |x - \bar{T}(x)|^2 \right) + o(\varepsilon) \\ &\geq C_p (x_d - \bar{s}) + o(\varepsilon). \end{aligned}$$

Notice that given $n_i \in \mathbb{N}$, to be fixed later, for any $x \in C^i_\varepsilon \setminus C_{\frac{\varepsilon}{n_i}}$ we have that $|x_d - \bar{s}| \geq \frac{\varepsilon}{n_i}$. Hence, integrating with respect to $\bar{\sigma}_i$ over C^i_ε yields

$$\begin{aligned} &\int_{C^i_\varepsilon} \left(|x - \mathrm{proj}_d(x)|^p - |x - \bar{T}(x)|^p \right) \mathrm{d}\bar{\sigma}_i \geq C_p \int_{C^i_\varepsilon \setminus C^i_{\frac{\varepsilon}{n_i}}} |x_d - \bar{s}| \mathrm{d}\bar{\sigma}_i + o(\varepsilon^2) \\ &\geq C_p \frac{\varepsilon}{n_i} \bar{\sigma}_i \left(C^i_\varepsilon \setminus C^i_{\frac{\varepsilon}{n_i}} \right) + o(\varepsilon^2) = C_p \frac{\varepsilon}{n_i} \left(\bar{\sigma}_i \left(C^i_\varepsilon \right) - \bar{\sigma}_i \left(C^i_{\frac{\varepsilon}{n_i}} \right) \right) + o(\varepsilon^2) \\ &\geq \frac{C_p}{n_i} \left(\frac{1}{\bar{\theta}_i} - \frac{\bar{\theta}_i}{n_i} \right) \varepsilon^2 + o(\varepsilon^2) \geq \frac{C_p}{2\bar{\theta}_i n_i} \varepsilon^2 + o(\varepsilon^2), \end{aligned}$$

where in the last inequality we choose $n_i \geq 2\bar{\theta}_i^2$.

For the indexes $i \notin I_2$, we observe that the error committed by using the map $\bar{T}$ is given by $|x - \mathrm{proj}_d(x)|^2 - |x - \bar{s}e_d|^2 = -(x_d - \bar{s})^2 \geq -\varepsilon^2$. So using once again (3.26) we get that

$$|x - \mathrm{proj}_d(x)|^p - |x - \bar{s}e_d|^p \geq -C_p \varepsilon^2 + o(\varepsilon^2).$$

Now setting $\nu' \stackrel{\text{def.}}{=} \bar{T}_\sharp \bar{\sigma}$, we obtain that

$$\begin{aligned} W_p^p(\bar{\sigma}, \bar{\nu}) - W_p^p(\bar{\sigma}, \nu') &\geq \int \left(|x - \mathrm{proj}_d(x)|^p - |x - \bar{T}(x)|^p \right) \mathrm{d}\bar{\sigma} \\ &= \sum_{i=1}^N \int_{C^i_\varepsilon} \left(|x - \mathrm{proj}_d(x)|^p - |x - \bar{T}(x)|^p \right) \mathrm{d}\bar{\sigma}_i \\ &\geq C_p \left(\sum_{i \in I_1} \left(\frac{1}{2\bar{\theta}_i n_i} \varepsilon^2 + o(\varepsilon^2) \right) - \sum_{i \in I_2} (\varepsilon^2 + o(\varepsilon^2)) \bar{\sigma}_i(C^i_\varepsilon) \right) \\ &= C_p \varepsilon^2 \left(\sum_{i \in I_1} \left(\frac{1}{2\bar{\theta}_i n_i} + \frac{o(\varepsilon^2)}{\varepsilon^2} \right) - \sum_{i \in I_2} \left(1 + \frac{o(\varepsilon^2)}{\varepsilon^2} \right) \bar{\sigma}_i(C^i_\varepsilon) \right). \end{aligned}$$

The last quantity must be positive for ε large enough since $\bar{\sigma}_i(C^i_\varepsilon) = o(\varepsilon)$, for each $i \in I_2$. But as the new competitor ν' is admissible for the minimization of F, we obtain a contradiction with the fact that $\bar{\nu}$ is a minimizer from Lemma 3.4. This contradicts the entire construction, meaning that Σ does not contain a loop.

Case 2: In the second case we only know that

$$\bar{\sigma} \left(T_{y_0} \Sigma \cap \overline{B_1(0)} \right) = 0.$$

Therefore, setting $D_\delta \overset{\text{def.}}{=} \{x : \operatorname{dist}(x, T_{y_0}\Sigma) \leq \delta\}$ we observe that

$$\bar{\sigma}(D_\delta) \xrightarrow[\delta \to 0^+]{} 0.$$

Next, we perform a similar construction from the one in the previous case, but this time we define

$$E_i \overset{\text{def.}}{=} \left\{x = (x', x_d) \in \mathbb{R}^d : \langle \xi_i, x\rangle > \delta/2,\ |x_d| \leq 1\right\} \setminus D_\delta,$$

where δ will be chosen later in order for the mass $\bar{\sigma}(D_\delta)$ to be small enough. As in Case 1 we can define

$$F_1 \overset{\text{def.}}{=} E_1, \quad F_i = E_i \setminus F_{i-1}, \quad F_0 \overset{\text{def.}}{=} \mathbb{R}^d \setminus \left(\bigcup_{i=1}^N F_i\right)$$

and the measures

$$\bar{\sigma}_i \overset{\text{def.}}{=} \bar{\sigma} \llcorner F_i, \quad \bar{\nu}_i \overset{\text{def.}}{=} [\operatorname{proj}_d]_\sharp \bar{\sigma}_i, \quad \text{for } i = 0, \ldots, N$$

so that in particular we have that $\bar{\sigma} = \sum_{i=0}^N \bar{\sigma}_i$ and $\bar{\nu} = \sum_{i=0}^N \bar{\nu}_i$. In particular, each $\bar{\nu}_i$ is rectifiable being written as $\bar{\nu}_i = \theta_i \mathscr{H}^1 \llcorner [-e_d, e_d]$, and it holds that $\sum_{i=0}^N \theta_i = \theta$.

One again, we consider a Lebesgue point $\bar{s}$ of all densities θ_i and ε_0 small enough so that for any $\varepsilon < \varepsilon_0$ the equivalent of (3.24) holds for all $i = 1, \ldots, N$. We also recall the sets of indexes I_1 and I_2 from (3.23), distinguishing the ones with positive density, $\theta_i > 0$ for $i \in I_1$ and $\theta_i = 0$ for $i \in I_2$. Given the value of ε we can choose δ small enough to have

$$\bar{\sigma}(D_\delta) \leq \varepsilon^2.$$

Finally, we construct the better competitor. For the indexes $i \in I_2$, we send all the mass of $\bar{\sigma}_i$ onto a Dirac mass concentrated at

$$y_\varepsilon \overset{\text{def.}}{=} (\bar{s} + \varepsilon) e_d, \text{ with total mass } m_\varepsilon \overset{\text{def.}}{=} \sum_{i \in I_2} \bar{\sigma}_i(C_\varepsilon^i).$$

But for $i \in I_1$, notice that $\delta = \delta(\varepsilon)$, and if for instance $\varepsilon \ll \delta$, we can proceed as in Case 1 and transport $\bar{\sigma}_i$ to a segment perpendicular to e_d, instead of transporting them to $\bar{\nu}_i$. As for the mass of $\bar{\sigma}_0$, we project it onto the newly added structure.

Since we do not have much information on the measures $\bar{\sigma}_i$ we cannot ensure this is the case. Instead, we let N_i be the smallest integer such that

$$\ell_i(\varepsilon) \overset{\text{def.}}{=} \frac{\alpha}{N_i} \bar{\sigma}_i(C_\varepsilon^i) \leq \frac{\delta}{2}.$$

Therefore, we can transport N_i copies of the measure $\bar{\sigma}_{i,N_i} = \dfrac{1}{N_i}\bar{\sigma}_i$ to the measures $\alpha^{-1}\mathscr{H}^1 \llcorner L_{i,j}$ uniformly distributed over the segments $L_{i,j} \overset{\text{def.}}{=} \bar{s} + [0, \ell_i(\varepsilon)]\xi_{i,j}$. Here $\left(\xi_{i,j}\right)_{j=1}^{N_i}$ are directions chosen in such a way that $L_{i,j}$ only intersect at their base point and such that $\left\langle \xi_i, \xi_{i,j}\right\rangle > 1/2$ for all $j = 1, \ldots, N_i$.

Defining $\Gamma \overset{\text{def.}}{=} \bigcup_{\substack{j=1,\dots,N_i \\ i=1,\dots,N}} L_{i,j}$, the new competitor then becomes

$$\nu' \overset{\text{def.}}{=} m_\varepsilon \delta_{y_\varepsilon} + \sum_{i=1}^{N} \sum_{j=1}^{N_i} \alpha^{-1} \mathscr{H}^1 \llcorner L_{i,j} + [\mathrm{proj}_\Gamma]_\sharp \bar{\nu}_0.$$

As a result, we can estimate the gain in transportation distance in a similarly to Case 1 as

$$W_p^p(\bar{\sigma}_0, \bar{\nu}_0) - W_p^p(\bar{\sigma}_0, \nu') \geq W_p^p(\bar{\sigma}_0, \bar{\nu}_0) + \sum_{i \in I_2} \int_{C_\varepsilon^i} \left\{ |x - y_\varepsilon|^p - |x - \mathrm{proj}_d(x)|^p \right\} \mathrm{d}\bar{\sigma}_i$$
$$+ \sum_{i \in I_1} \sum_{j=1}^{N_i} \left\{ W_p^p \left(\frac{1}{N_i} \bar{\sigma}_i \llcorner C_\varepsilon^i, \alpha^{-1} \mathscr{H}^1 \llcorner L_{i,j} \right) - \frac{1}{N} W_p^p \left(\bar{\sigma}_i \llcorner C_\varepsilon^i, \bar{\nu}_i \llcorner C_\varepsilon^i \right) \right\}.$$

The first term is a $o(\varepsilon^2)$ since

$$W_p^p(\bar{\sigma}_0, \bar{\nu}_0) \leq \delta^p \bar{\sigma}_0(D_\delta) \leq \delta^p \varepsilon^2,$$

and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. The same is true for the first sum, since by definition for all $i \in I_2$ it holds that $\bar{\sigma}_i(C_\varepsilon^i) = o(\varepsilon)$ and $p \geq 1$. Finally, by the estimations done in Case 1, each of the terms inside the double sum are bounded from below by a term of the form $C\varepsilon^2$ with $C > 0$.

Summing up all these contributions, we obtain the bound

$$W_p^p(\bar{\sigma}_0, \bar{\nu}_0) - W_p^p(\bar{\sigma}_0, \nu') \geq C\varepsilon^2 + o(\varepsilon^2),$$

but as by construction the new competitor ν' satisfies the constraints of F, we see that for ε small enough it strictly improves its value. This contradicts the minimality of $\bar{\nu}$ and the entire construction, meaning that Σ can not have loops in Case 2 either. ☐

Appendix A. Appendix: technical proofs of the localization/blow-up argument

In this appendix we give the technical proofs of Lemma 3.2 and Thm. 3.3, which are strongly inspired on the arguments from [5]. We recall that as throughout Section 3 $\nu_\star$ is a fixed minimizer of the relaxed problem and $\alpha \overset{\text{def.}}{=} \mathcal{L}(\nu_\star)$.

Lemma A.1. *The localized measure ν_n solves the following minimization problem*

$$\min \left\{ W_p^p(\sigma_n, \nu') : \begin{array}{c} \text{there is } \Sigma' \in \mathcal{A}_2 \text{ such that} \\ \nu' \in \mathscr{M}_+(\Sigma'),\ \nu' \geq \alpha^{-1} \mathscr{H}^1 \llcorner \Sigma', \\ \Sigma_n \cup \Sigma' \text{ is connected}, \\ \nu'(\overline{B_1(0)}) = \nu_\star\left(\Sigma_{y_0, r_n}\right) \end{array} \right\}. \tag{A.1}$$

Proof. Let γ be the optimal transportation plan between ϱ_0 and $\nu_\star$. Recall the notation

$$\Sigma_{y_0, r_n} \overset{\text{def.}}{=} \Sigma \cap B_{r_n}(y_0) \text{ and } \Sigma_n \overset{\text{def.}}{=} \Sigma \setminus B_{r_n}(y_0).$$

By construction both sets are connected, and define the new transportation plan

$$\tilde{\gamma} \overset{\text{def.}}{=} \gamma \llcorner \mathbb{R}^d \times \Sigma_n + \gamma',$$

where γ' is optimal between ϱ_n and ν'. Then the new competitor $\tilde{\nu} \overset{\text{def.}}{=} (\pi_1)_\sharp \tilde{\gamma}$ is such that $\mathcal{L}(\tilde{\nu}) \leq \mathcal{L}(\nu_\star)$, and the optimality of $\nu_\star$ gives that

$$\int_{\mathbb{R}^d \times \Sigma_n} |x-y|^p \mathrm{d}\gamma + \int_{\mathbb{R}^d \times \Sigma_{y_0,r_n}} |x-y|^p \mathrm{d}\gamma \leq \int_{\mathbb{R}^d \times \Sigma_n} |x-y|^p \mathrm{d}\gamma + \int |x-y|^p \mathrm{d}\gamma'$$

Giving that $W_p^p(\varrho_n, \nu_n) \leq W_p^p(\varrho_n, \nu')$ for all ν' admissible.

But we need to test the optimality of ν_n for the transport with initial measure given by σ_n. The latter was constructed to be a geodesic interpolation between ϱ_n and ν_n, see for instance [15, Thm. 5.27]. As such, it holds that

$$\begin{aligned} W_p(\varrho_n, \sigma_n) + W_p(\sigma_n, \nu_n) &= W_p(\varrho_n, \nu_n) \\ &\leq W_p(\varrho_n, \nu') \leq W_p(\varrho_n, \sigma_n) + W_p(\sigma_n, \nu'), \end{aligned}$$

where above we have used the optimality of ν_n for the transport with ϱ_n and the triangle inequality. Canceling the terms $W_p(\varrho_n, \sigma_n)$ the result follows. $\square$

In the sequel, we prove Thm. 3.3. In fact, problem (A.1), and consequently the functionals F_n and F, have been modified from their counterparts in [5] in order to simplify the Γ-convergence result that follows. Whereas the formulation in [5] was chosen to be as general as possible; here we intend to show how we can facilitate greatly this proof by considering perturbations that are connected.

Theorem A.2. *The family $(F_n)_{n\in\mathbb{N}}$ converges to F in the sense of Γ-convergence, for the topology of weak-$\star$ convergence of Radon measures.*

Proof. Let us start with the $\Gamma - \liminf$, so consider a sequence $(\nu'_n)_{n\in\mathbb{N}}$ converging in the narrow topology to ν', and such that $\liminf_{n\to\infty} F_n(\nu'_n) < +\infty$, so we can assume that for each $n \in \mathbb{N}$ there is a set Σ'_n such at most 2 connected components such that

$$\Sigma'_n = \operatorname{supp} \nu'_n, \quad \Sigma'_n \subset \overline{B_1(0)}, \quad \alpha \nu'_n \geq \mathscr{H}^1 \llcorner \Sigma'_n.$$

Since $\Sigma'_n \subset \overline{B_1(0)}$, we can apply Blaschke's Theorem assuming that $\Sigma'_n \xrightarrow[n\to\infty]{d_H} \Sigma'$, up to a not relabelled subsequence. The limit Σ' also has at most 2 connected components; and applying Gołąb's Theorem to ν' restricted to each connected component it holds that

$$\alpha \nu' \geq \mathscr{H}^1 \llcorner \Sigma' \text{ and } \nu' \in \mathscr{M}_+(\Sigma').$$

In addition, recall that by the construction from Lemma 2.5

$$\frac{\Sigma_n - y_0}{r_n} \cap \partial B_1(0) = \{y_{1,n}, y_{2,n}\} = \frac{\Sigma_{y_0,r_n} - y_0}{r_n} \cap \partial B_1(0).$$

Since $\dfrac{\Sigma_{y_0,r_n} - y_0}{r_n}$ converges to $[-\tau, \tau]$ we must have that $y_{i,n} \xrightarrow[n\to\infty]{} (-1)^i \tau$ for $i = 1,2$. But since $\Sigma'_n \subset \overline{B_1(0)}$, the only way it is connected to $\dfrac{\Sigma_n - y_0}{r_n}$ is if it contains at least one of $y_{i,n}$, or both if it has two connected components. We then conclude that at least one of $-\tau,\tau$ belong to Σ'.

As a result, ν' is in the domain of F and from the lower semi-continuity of the Wasserstein distance we get that

$$F(\nu') = W_p^p(\bar{\sigma},\nu') \le \liminf_{n\to\infty} W_p^p(\bar{\sigma}_n,\nu'_n) = \liminf_{n\to\infty} F_n(\nu'_n).$$

Γ-limsup: The strategy to prove the limsup is based on three steps: first we renormalize ν' to satisfy the mass constraint in F_n, which may break the condition $\alpha\nu'_n \ge \mathscr{H}^1 \llcorner \Sigma'$, so we shrink the support to satisfy it again. Assuming that Σ' has two connected components Σ'_1, Σ'_2, we translate the mass of each of their shrunk versions so that it is connected to $\dfrac{\Sigma_n - y_0}{r_n}$. Since some parts of the support may get out of $\overline{B_1(0)}$, we project the residual mass onto $\overline{B_1(0)}$.

Let us construct a recovery sequence $(\nu'_n)_{n\in\mathbb{N}}$. By the constraint that $T_{y_0}\Sigma \cap \partial B_1(0) \subset \Sigma'$, the unit vectors $\pm\tau$ must be contained in each of the connected components Σ'_1, Σ'_2. It is also possible that one of them is just a singleton $\pm\tau$ and only the other has positive length, or that Σ' has only one connected component which contains both, but the following argument works for both cases with straightforward adaptations. By the Kuratowski (even Hausdorff) convergence of Σ_{y_0,r_n} towards $[-\tau,\tau]$, for each $i \in I$, there exists a sequence $(y_{i,n})_{n\in\mathbb{N}}$ such that $y_{i,n} \in \Sigma_{y_0,r_n}$ for each $n \in \mathbb{N}$, and $y_{i,n} \to (-1)^i\tau$. We then define:

$$a_n \overset{\text{def.}}{=} \frac{\nu_\star(\Sigma_{y_0,r_n})}{2r_n\theta_1(\nu_\star, y_0)}, \quad \text{and} \quad s_n \overset{\text{def.}}{=} \max(1, a_n^{-1}),$$

noting that $a_n \to 1$ and $s_n \to 1$, and we introduce the map T_n,

$$T_n(y) \overset{\text{def.}}{=} (y + (-1)^{i+1}\tau)/s_n + y_{i,n}, \text{ if } y \in \Sigma'_i$$

The map T_n shrinks each connected component Σ'_i and translates it to the corresponding $y_{i,n} \in \Sigma_{y_0,r_n}$. It follows that

$$\frac{\Sigma_n - y_0}{r_n} \cup T_n\left(\Sigma'\right)$$

is connected, but not necessarily contained in $\overline{B_1}$; so we project it onto it and preserve connectedness. To perform this operation, let proj_{B_1} denote the projection onto the closed unit ball and define

$$\nu'_n \overset{\text{def.}}{=} (\mathrm{proj}_{B_1} \circ T_n)_\sharp(a_n\nu') \text{ and } \Sigma'_n \overset{\text{def.}}{=} (\mathrm{proj}_{B_1} \circ T_n)(\Sigma').$$

Let us check that ν'_n converges to ν' in the narrow topology. For $y \in \Sigma'_i$,

$$|y - T_n(y)| \le |y|(1 - 1/s_n) + |(-1)^{i+1}/s_n - y_{i,n}| \xrightarrow[n\to+\infty]{} 0.$$

By the dominated convergence theorem, we get that for any $\phi \in \mathscr{C}_b(\mathbb{R}^d)$,

$$\int \phi \mathrm{d}\nu'_n = a_n \int_{\Sigma'} \phi\left(\mathrm{proj}_{B_1} \circ T_n(y)\right) \mathrm{d}\nu'(y) \xrightarrow[n\to+\infty]{} \int_{\Sigma'} \phi\left(y\right) \mathrm{d}\nu'(y)$$

so that $\nu'_n \xrightarrow[n\to+\infty]{} \nu'$ in the narrow topology.

Let us now check the constraints in F_n. From the properties of image measures, we see that the mass of ν_n' is concentrated in $\Sigma_n' \subset \overline{B_1(0)}$ which is such that

$$\frac{\Sigma_n - y_0}{r_n} \cup \Sigma_n',$$

is connected by the previous arguments, and we also have

$$\nu_n'(\overline{B_1(0)}) = a_n \nu'(\mathbb{R}^d) = \frac{\nu_\star(\Sigma_{y_0,r_n})}{r_n}$$

so that ν_n' has the mass prescribed by F_n.

It only remains to show that is satisfies the density constraints, take any non-negative $\phi \in \mathscr{C}_b(\mathbb{R}^d)$,

$$\begin{aligned}
\alpha \int_{\mathbb{R}^d} \phi \mathrm{d}\nu_n' &= \alpha a_n \int_{\Sigma'} \phi\left(\mathrm{proj}_{B_1} \circ T_n(y)\right) \mathrm{d}\nu'(y) \\
&\geq a_n \int_{\Sigma'} \phi\left(\mathrm{proj}_{B_1} \circ T_n(y)\right) \mathrm{d}\mathscr{H}^1(y) \\
&= a_n s_n \int_{\Sigma_n'} \phi\left(\mathrm{proj}_{B_1} \circ T_n(y')\right) \mathrm{d}\mathscr{H}^1(y') \geq \int_{\Sigma_n'} \phi \mathrm{d}\mathscr{H}^1.
\end{aligned}$$

It follows that $\alpha\nu_n' \geq \mathscr{H}^1 \llcorner \Sigma_n'$ and we conclude that $F_n(\nu_n) < \infty$, for all $n \in \mathbb{N}$.

By the continuity of the Wasserstein distance with respect to the narrow convergence (provided the measures are supported in some common compact set), we have that:

$$F_n(\nu_n') \xrightarrow[n\to\infty]{} F(\nu').$$

The Γ-convergence follows.

□

References

[1] Luigi Ambrosio, Elia Brué, and Daniele Semola. Lectures on optimal transport, 2021.

[2] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Courier Corporation, 2000.

[3] Marcus Brazil, Ronald L Graham, Doreen A Thomas, and Martin Zachariasen. On the history of the euclidean steiner tree problem. *Archive for history of exact sciences*, 68(3):327–354, 2014.

[4] Giuseppe Buttazzo and Eugene Stepanov. Optimal transportation networks as free dirichlet regions for the monge-kantorovich problem. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 2(4):631–678, 2003.

[5] Antonin Chambolle, Vincent Duval, and João Miguel Machado. One dimensional approximation of measures in Wasserstein distance. *Journal de l'École Polytechnique-Mathématiques*, 12:101–145, 2025.

[6] Camillo De Lellis. Lecture notes on rectifiable sets, densities, and tangent measures. *Preprint*, 23, 2006.

[7] Antonie Lemenant. A presentation of the average distance minimizing problem. *Journal of Mathematical Sciences*, 181(6), 2012.

[8] João Miguel Machado. Phase-field approximation for 1-dimensional shape optimization problems. *to appear in SIAM Journal of Mathematical Analysis*, 2025.

[9] Francesco Maggi. *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory*. Number 135. Cambridge University Press, 2012.

[10] Sunra JN Mosconi, Paolo Tilli, et al. Γ-convergence for the irrigation problem. *J. Convex Anal*, 12(1):145–158, 2005.

[11] Lucas O'Brien, Forest Kobayashi, and Young-Heon Kim. Structure of average distance minimizers in general dimensions. *arXiv preprint arXiv:2503.23256*, 2025.

[12] Emanuele Paolini and Eugene Stepanov. Qualitative properties of maximum distance minimizers and average distance minimizers in rn. *Journal of Mathematical Sciences*, 122(3):3290–3309, 2004.

[13] Emanuele Paolini and Eugene Stepanov. Existence and regularity results for the steiner problem. *Calculus of Variations and Partial Differential Equations*, 46(3):837–860, 2013.

[14] R Tyrrell Rockafellar and Roger J-B Wets. *Variational analysis*, volume 317. Springer Science & Business Media, 2009.

[15] F. Santambrogio. Optimal transport for applied mathematicians. *Birkäuser, NY*, 55(58-63):94, 2015.

[16] Filippo Santambrogio and Paolo Tilli. Blow-up of optimal sets in the irrigation problem. *The Journal of Geometric Analysis*, 15:343–362, 2005.

[17] Cédric Villani. *Optimal transport: old and new*, volume 338. Springer, 2009.

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